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On the rth dispersionless Toda hierarchy: factorization problem, additional symmetries and some solutions

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Abstract

For a family of Poisson algebras, parametrized by $r \in \mathbb{Z}$, and an associated Lie algebraic splitting, we consider the factorization of given canonical transformations. In this context, we rederive the recently found rth dispersionless modified KP hierarchies and rth dispersionless Dym hierarchies, giving a new Miura map among them. We also found a new integrable hierarchy which we call the rth dispersionless Toda hierarchy. Moreover, additional symmetries for these hierarchies are studied in detail and new symmetries depending on arbitrary functions are explicitly constructed for the rth dispersionless KP, rth dispersionless Dym and rth dispersionless Toda equations. Some solutions are derived by examining the imposition of a time invariance to the potential rth dispersionless Dym equation, for which a complete integral is presented and, therefore, an appropriate envelope leads to a general solution. Symmetries and Miura maps are applied to get new solutions and solutions of the rth dispersionless modified KP equation.

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1. Introduction

The study of dispersionless integrable hierarchies is a subject of increasing activity in the theory of integrable systems. This was originated in several sources; we mention here the pioneering work of Kodama and Gibbons [1] on the dispersionless KP, of Kupershimdt on the dispersionless modified KP [2] and the role of Riemann invariants and hodograph transformations found by Tsarev [3]. The important work of Takasaki and Takabe [4–6] gave the Lax formalism, additional symmetries, twistor formulation of the dispersionless KP and dispersionless Toda hierarchies; see [7] for the dispersionless Dym (or Harry Dym) equation.

The appearance of dispersionless systems in topological field theories is also an important issue, see [8, 9]. More recent progress appears in relation with the theory of conformal maps [10, 11], quasiconformal maps and $\bar{\partial}$-formulation [12], reductions of several type [13, 14], additional symmetries [15] and twistor equations [16], on hodograph equations for the Boyer–Finley equation [17] and its applications in general relativity; see also [18, 19]. The approach given in [20] to the theory is also remarkable. Finally, we comment on the contribution to electrodynamics and the dispersionless Veselov–Novikov equation [21].

Recently, a new Poisson bracket and the associated Lie algebra splitting, therefore using a Lax formalism and an r-matrix approach, were presented in [22] to construct new dispersionless integrable hierarchies and, later on (see [23]), the theory was further extended. We must remark that, since the work of Golenischeva and Rieman [24] and Li [7], the possibility for a r-matrix formulation of dispersionless integrable systems was known.

In this paper we shall use this new splitting together with a standard technique in the theory of integrable hierarchies, the factorization problem, to get a new dispersionless integrable hierarchy, which we call the rth dispersionless Toda hierarchy (for $r = 1$ we get the well-known dispersionless Toda hierarchy). The use of the factorization problem allows us an analysis closer to the dressing group technique of Segal–Wilson [25]. For that aim we consider the Lie group of canonical transformations associated with a particular Poisson bracket together with a factorization problem induced by a r-matrix associated with a canonical splitting of the corresponding algebra of symplectic vector fields [22]. These new hierarchies contain dispersionless integrable equations derived previously in [22]—which we call rth dispersionless modified KP and rth dispersionless Dym equations (for $r = 0$ we obtain the well-known dispersionless modified KP and dispersionless Dym equations).

In [26] the Miura map among the dispersionless modified KP and dispersionless Dym equations was presented; here we extend those results to the present context. We also study in detail the additional or master symmetries of these hierarchies; in particular for the three integrable equations, rth dispersionless modified KP, rth dispersionless Dym and rth dispersionless Toda equations. We get new explicit symmetries depending on arbitrary functions, in the spirit of the symmetries given in [18] for the dispersionless KP equation.

Later we find a complete integral for the t_2-reduction of the potential rth dispersionless Dym equation. Thus, using the method of the complete solution, an appropriate envelope leads to its general solution. Then, when the symmetries are applied we obtain more general solutions, non-t_2 invariant of the potential rth dispersionless Dym equation. Using the Miura map, new solutions of the rth dispersionless modified KP equation are obtain and the corresponding functional symmetries are applied to get more general families of solutions. Finally, we also derive from the factorization problem twistor equations for these integrable hierarchies.

We have developed, in [27] the hodograph reduction technique to get ample families of implicit and explicit solutions to the r-dmKP and r-dDym hierarchies.

The layout of this paper is as follows. In section 2 we consider the Lie algebra setting, the factorization problem and its differential description; Lax functions are given as well. Then, in section 3, the corresponding integrable hierarchies are derived together with the Miura map. Next, in section 4, we introduce Orlov functions and show how the factorization problem constitutes a simple framework to derive the corresponding additional symmetries of the integrable hierarchies. In particular, additional symmetries (which depend on arbitrary functions of t_2) of the integrable dispersionless equations discussed in section 3 are found. Finally, in section 5, some solutions of these integrable hierarchies and section 5 a twistor formulation of these dispersionless integrable hierarchies derived from the factorization problem are given. In a forthcoming paper we will give a twistor formulation of these hierarchies which is derived independently of the factorization problem.

2. Factorization problem and its differential versions

We shall work with the Lie algebra $\mathfrak{g}$ of Laurent series $H(p,x) := \sum_{n\in\mathbb{Z}} u_n(x)p^n$ in the variable $p \in \mathbb{R}$ with coefficients depending on the variable $x \in \mathbb{R}$, with Lie commutator given by the following Poisson bracket [22]:

$$\{H_1, H_2\} = p^r\left(\frac{\partial H_1}{\partial p}\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial x}\frac{\partial H_2}{\partial p}\right), \quad r \in \mathbb{Z}.$$

Observe that for each $r \in \mathbb{Z}$ we are dealing with a different Lie algebra; notice also that this Poisson bracket is associated with the the following sympletic form:

$$\omega := p^{-r}\,\mathrm{d}p \wedge \mathrm{d}x.$$

2.1. The Lie algebra splitting

We shall use the following triangular type splitting of $\mathfrak{g}$ into Lie subalgebras:

$$\mathfrak{g} = \mathfrak{g}_> \oplus \mathfrak{g}_{1-r} \oplus \mathfrak{g}_<, \tag{1}$$

where

$$\mathfrak{g}_{\gtrless} := \mathbb{C}\{u_n(x)p^n\}_{n\gtrless(1-r)}, \qquad \mathfrak{g}_{1-r} := \mathbb{C}\{u(x)p^{1-r}\},$$

and fulfil the following property:

$$\{\mathfrak{g}_{\gtrless}, \mathfrak{g}_{1-r}\} = \mathfrak{g}_{\gtrless}.$$

If we define the Lie subalgebra $\mathfrak{g}_{\geqslant}$ as

$$\mathfrak{g}_{\geqslant} := \mathfrak{g}_{1-r} \oplus \mathfrak{g}_>,$$

we have the direct sum decomposition of the Lie algebra $\mathfrak{g}$ given by

$$\mathfrak{g} = \mathfrak{g}_< \oplus \mathfrak{g}_{\geqslant}. \tag{2}$$

We remark that allowing the variable p to take values in $\mathbb{C}$ we have the following interpretation for the above splitting (2). Suppose that $\mathfrak{g}$ is the set of analytic functions in some annulus of $p=0$. Then, $H\in\mathfrak{g}_<$ iff $H(p)p^{-1+r}$ is an analytic function outside the annulus which vanishes at $p=\infty$. On the contrary, a function $H \in \mathfrak{g}_{\geqslant}$ if $H(p)p^{-1+r}$ has an analytic extension inside the annulus.

Observe that the induced Lie commutator in $\mathfrak{g}_{1-r}$ is

$$\{f(x)p^{1-r}, g(x)p^{1-r}\} = (1-r)W(f,g)p^{1-r},$$

where $W(f,g) := fg_x - gf_x$ is the Wrońskian of f and g. Only when $r = 1$ the Lie subalgebra $\mathfrak{g}_{1-r}$ is an Abelian Lie subalgebra.

An alternative realization of $\mathfrak{g}$ is through the adjoint action

$$\mathrm{ad}: \mathfrak{g} \to \mathfrak{X}(\mathbb{R}^2), \qquad H \mapsto \mathrm{ad}_H := \{H, \cdot\},$$

so that

$$\mathrm{ad}_H = p^r\frac{\partial H}{\partial p}\frac{\partial}{\partial x} - p^r\frac{\partial H}{\partial x}\frac{\partial}{\partial p}.$$

2.2. The factorization problem

Given an element $H \in \mathfrak{g}$, the corresponding vector field ad_H generates a symplectic diffeormorphism (canonical transformation) Φ_H, given by

$$\Phi_H := \sum_{n=0}^{\infty} \frac{1}{n!} (\mathrm{ad}_H)^n.$$

This transformation corresponds to the element $h = \exp(H)$ belonging to the local Lie group G generated by $\mathfrak{g}$. In fact, $\Phi_H = \mathrm{Ad}_h$, the adjoint action of the Lie group G on $\mathfrak{g}$.

In what follows, we shall denote by $G_<, G_{1-r}, G_>$ and $G_\geqslant$ the local Lie groups corresponding to the Lie algebras $\mathfrak{g}_<, \mathfrak{g}_{1-r}, \mathfrak{g}_>$ and $\mathfrak{g}_\geqslant$, respectively.

Given $h, \bar{h} \in G$ in the local Lie group G, the finding of $h_< \in G_<$ and $h_\geqslant \in G_\geqslant$ such that the following factorization holds

$$h_< \cdot h = h_\geqslant \cdot \bar{h}, \tag{3}$$

will play a pivotal role in what follows.

Furthermore, given two sets of deformation parameters $(t_1, t_2, \ldots)$ and $(\bar{t}_1, \bar{t}_2, \ldots)$ and corresponding elements in the Lie algebra $\mathfrak{g}$

$$t(p) := t_1 p^{2-r} + t_2 p^{3-r} + \cdots \in \mathfrak{g}_>, \qquad \bar{t}(p) := \bar{t}_1 p^{-r} + \bar{t}_2 p^{-r-1} + \cdots \in \mathfrak{g}_<,$$

we shall analyse the following deformation of (3):

$$\psi_< \cdot \exp(t) \cdot h = \psi_\geqslant \cdot \exp(\bar{t}) \cdot \bar{h}. \tag{4}$$

Notice that there is no loss of generality if we set $\bar{h} = 1$ in (4) so that

$$\exp(t) \cdot h \cdot \exp(-\bar{t}) = \psi_<^{-1} \cdot \psi_\geqslant. \tag{5}$$

2.3. Differential consequences of the factorization problem

A possible way to study (4) is by analysing its differential versions, i.e. by taking right derivatives.

Given a derivation ∂ of a Lie algebra $\mathfrak{g}$, see [4, 28], one defines the corresponding right derivative in the associated local Lie group by

$$\partial h \cdot h^{-1} := \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\mathrm{ad}\, H)^n (\partial H), \qquad h := \exp(H), \quad H \in \mathfrak{g}. \tag{6}$$

Observe that, in [4], one has the notation $\nabla_H H = \partial h \cdot h^{-1}$, while in [28] the notation ∂h is used. We have the two important equations [4]

$$\partial(h_1 h_2) \cdot (h_1 h_2)^{-1} = \partial h_1 \cdot h_2^{-1} + \mathrm{Ad}\, h_1 (\partial h_2 \cdot h_2^{-1})$$

$$\partial(h_1^{-1}) \cdot (h_1^{-1})^{-1} = -\mathrm{Ad}(h_1^{-1})(\partial h_1 \cdot h_1^{-1}).$$

Hence, by taking right derivatives of (4) with respect to

$$\partial_n := \frac{\partial}{\partial t_n}, \qquad \bar{\partial}_n := \frac{\partial}{\partial \bar{t}_n},$$

we get

$$\partial_n \psi_< \cdot \psi_<^{-1} + \mathrm{Ad}_{\psi_<}(p^{n+1-r}) = \partial_n \psi_\geqslant \cdot \psi_\geqslant^{-1}, \tag{7}$$

$$\bar{\partial}_n \psi_< \cdot \psi_<^{-1} = \bar{\partial}_n \psi_\geqslant \cdot \psi_\geqslant^{-1} + \mathrm{Ad}_{\psi_\geqslant}(p^{1-r-n}). \tag{8}$$

If we further factorize

$$\psi_\geqslant = \psi_{1-r} \cdot \psi_>,$$

equation (7) decomposes, according to (1), into the following three equations:

$$\partial_n \psi_< \cdot \psi_<^{-1} + P_< \mathrm{Ad}_{\psi_<} p^{n+1-r} = 0, \tag{9}$$

$$P_{1-r} \mathrm{Ad}_{\psi_<} p^{n+1-r} = \partial_n \psi_{1-r} \cdot \psi_{1-r}^{-1}, \tag{10}$$

$$P_> \mathrm{Ad}_{\psi_<} p^{n+1-r} = \mathrm{Ad}_{\psi_{1-r}}(\partial_n \psi_> \cdot \psi_>^{-1}). \tag{11}$$

Similar considerations applied to (8) lead to

$$\bar{\partial}_n \psi_< \cdot \psi_<^{-1} = \mathrm{Ad}_{\psi_{1-r}} P_< \mathrm{Ad}_{\psi_>} p^{1-r-n}, \tag{12}$$

$$0 = \bar{\partial}_n \psi_{1-r} \cdot \psi_{1-r}^{-1} + \mathrm{Ad}_{\psi_{1-r}} P_{1-r} \mathrm{Ad}_{\psi_>} p^{1-r-n}, \tag{13}$$

$$0 = \bar{\partial}_n \psi_> \cdot \psi_>^{-1} + P_> \mathrm{Ad}_{\psi_>} p^{1-r-n}. \tag{14}$$

Now we show that we can interchange the roles of $\mathfrak{g}_<$ and $\mathfrak{g}_\geqslant$. For this aim we introduce the map

$$F : \mathbb{R}^2 \to \mathbb{R}^2, \qquad (p, x) \mapsto (p' := 1/p, x' = -x), \tag{15}$$

which together with $r' = 2 - r$ may be considered as a "canonical transformation in the sense that the canonical form $\Omega = p^{-r}\,\mathrm{d}p \wedge \mathrm{d}x$ transforms onto

$$\Omega' = (p')^{-r'} \mathrm{d}p' \wedge \mathrm{d}x'.$$

If we denote

$$\mathfrak{g}'_\geqslant := \mathbb{C}\{u_n(x)(p')^n\}_{n \geqslant (1-r')}, \qquad \mathfrak{g}'_{1-r'} := \mathbb{C}\{u(x) p^{1-r'}\},$$

then

$$\mathfrak{g}_> \to \mathfrak{g}'_<, \qquad \mathfrak{g}_{1-r} \to \mathfrak{g}'_{1-r'}, \qquad \mathfrak{g}_< \to \mathfrak{g}'_>.$$

Observe that

$$t(p) = t_1 p^{2-r} + t_2 p^{3-r} + \cdots \to t_1 (p')^{-r'} + t_2 (p')^{-r'-1} + \cdots = \bar{t}'(p'),$$

$$\bar{t}(p) = \bar{t}_1 p^{-r} + \bar{t}_2 p^{-r-1} + \cdots \to \bar{t}_1 (p')^{2-r'} + \bar{t}_2 (p')^{3-r'} + \cdots = t'(p').$$

Thus, $t'_n = \bar{t}_n$ and $\bar{t}'_n = t_n$. Finally, the factorization (4) transforms as

$$\psi_< \cdot \exp(t) \cdot h = \psi_{1-r} \cdot \psi_> \cdot \exp(\bar{t}) \cdot \bar{h} \to \psi'^{-1}_{1-r'} \cdot \psi'_> \cdot \exp(\bar{t}') \cdot h = \psi'_< \cdot \exp(\bar{t}) \cdot \bar{h}. \tag{16}$$

With these observations is easy to see that our map transforms equations (9), (10) and (11) into (14), (13) and (12), respectively.

2.4. Lax formalism

Equations (9)–(14) can be given in a Lax form; for that aim we first introduce the following Lax functions:

$$L := \mathrm{Ad}_{\psi_<} p, \qquad \bar{\ell} := \mathrm{Ad}_{\psi_>} p, \qquad \bar{L} := \mathrm{Ad}_{\psi_{1-r}} \bar{\ell} = \mathrm{Ad}_{\psi_\geqslant} p \tag{17}$$

in terms of which equations (7) and (8) read as

$$\partial_n \psi_< \cdot \psi_<^{-1} = \partial_n \psi_\geqslant \cdot \psi_\geqslant^{-1} - L^{n+1-r}, \qquad \bar{\partial}_n \psi_\geqslant \cdot \psi_\geqslant^{-1} = \bar{\partial}_n \psi_< \cdot \psi_<^{-1} - \bar{L}^{1-r-n},$$

so that

$$\begin{aligned} \partial_n \psi_< \cdot \psi_<^{-1} &= -P_< L^{n+1-r}, \qquad & \partial_n \psi_\geqslant \cdot \psi_\geqslant^{-1} &= P_\geqslant L^{n+1-r}, \\ \bar{\partial}_n \psi_< \cdot \psi_<^{-1} &= P_< \bar{L}^{1-r-n}, \qquad & \bar{\partial}_n \psi_\geqslant \cdot \psi_\geqslant^{-1} &= -P_\geqslant \bar{L}^{1-r-n}, \end{aligned} \tag{18}$$

and therefore the following Lax equations hold:

$$\begin{aligned} \partial_n L &= \{-P_< L^{n+1-r}, L\}, \qquad & \partial_n \bar{L} &= \{P_\geqslant L^{n+1-r}, \bar{L}\}, \\ \bar{\partial}_n L &= \{P_< \bar{L}^{1-r-n}, L\}, \qquad & \bar{\partial}_n \bar{L} &= \{-P_\geqslant \bar{L}^{1-r-n}, \bar{L}\}. \end{aligned} \tag{19}$$

To deduce these equations just recall that if $B = \mathrm{Ad}_\phi b$, and ∂ is a Lie algebra derivation, then $\partial B = \{\partial\phi \cdot \phi^{-1}, B\} + \mathrm{Ad}_\phi \partial b$.

For a further analysis (18) is essential to get the powers of L and $\bar{L}$. In the following proposition we shall show how the powers of the Lax functions are connected with $\Psi_<$, $\Psi_>$ and ξ, the infinitesimal generators of $\psi_<$, $\psi_>$ and ψ_{1-r}, respectively,

$$\begin{aligned} \psi_< &:= \exp(\Psi_<), \qquad & \Psi_< &:= \Psi_1 p^{-r} + \Psi_2 p^{-r-1} + \cdots, \\ \psi_> &:= \exp(\Psi_>), \qquad & \Psi_> &:= \bar{\Psi}_1 p^{2-r} + \bar{\Psi}_2 p^{3-r} + \cdots, \\ \psi_{1-r} &:= \exp(\xi p^{1-r}). \end{aligned} \tag{20}$$

Proposition 1. *We can parametrize L^m, $\bar{\ell}^m$ and $\bar{L}^m$ in terms of Ψ_n, $\bar{\Psi}_n$ and ξ as follows:*

$$L^m = p^m + u_{m,0} p^{m-1} + u_{m,1} p^{m-2} + u_{m,2} p^{m-3} + \mathrm{O}(p^{m-4}), \qquad p \to \infty \tag{21}$$

$$\bar{\ell}^m = p^m + \bar{v}_{m,0} p^{m+1} + \bar{v}_{m,1} p^{m+2} + \bar{v}_{m,2} p^{m+3} + \mathrm{O}(p^{m+4}), \qquad p \to 0, \tag{22}$$

$$\bar{L}^m = \bar{u}_{m,-1} p^m + \bar{u}_{m,0} p^{m+1} + \bar{u}_{m,1} p^{m+2} + \bar{u}_{m,2} p^{m+3} + \mathrm{O}(p^{m+4}), \qquad p \to 0, \tag{23}$$

where the first coefficients are

$$\begin{aligned} u_{m,0} &= -m\Psi_{1,x}, \\ u_{m,1} &= m\left(-\Psi_{2,x} + \tfrac{1}{2}(r\Psi_1\Psi_{1,xx} + (m-1)\Psi_{1,x}^2)\right), \\ u_{m,2} &= m\left(-\Psi_{3,x} + \tfrac{1}{2}((r+1)\Psi_2\Psi_{1,xx} + (2m-3)\Psi_{1,x}\Psi_{2,x} + r\Psi_1\Psi_{2,xx})\right. \\ &\quad \left. -\tfrac{1}{6}(r^2\Psi_1^2\Psi_{1,xxx} + r(r+3m-4)\Psi_1\Psi_{1,x}\Psi_{1,xx} + (m-1)(m-2)\Psi_{1,x}^3)\right), \end{aligned} \tag{24}$$

$$\begin{aligned}
\bar v_{m,0} &= -m\bar\Psi_{1,x},\\
\bar v_{m,1} &= m(-\bar\Psi_{2,x} - \tfrac12((2-r)\bar\Psi_1\bar\Psi_{1,xx} - (m+1)\bar\Psi_x^2),\\
\bar v_{m,2} &= m(-\bar\Psi_{3,x} - \tfrac12((3-r)\bar\Psi_2\bar\Psi_{1,xx} - (2m+3)\Psi_{1,x}\Psi_{2,x} + (2-r)\bar\Psi_1\bar\Psi_{2,xx})\\
&\quad -\tfrac16((2-r)^2\bar\Psi_1^2\bar\Psi_{1,xxx} + (2-r)(r+3m+2)\bar\Psi_1\bar\Psi_{1,x}\bar\Psi_{1,xx} + (m+1)(m+2)\bar\Psi_{1,x}^3)),
\end{aligned}\tag{25}$$

and

$$\bar u_{m,j} = \begin{cases} \left(\dfrac{\xi\circ X}{\xi}\right)^{-(m+1+j)/(1-r)} \bar v_{m,j}\circ X, \quad \displaystyle\int_x^X \frac{\mathrm{d}x}{\xi(x)} = 1-r, & r\neq 1,\\ \exp(-(m+1+j)\xi_x)\bar v_{m,j}, & r = 1.\end{cases}$$

Proof. To evaluate L^m we recall that L^m is connected to p^m through a canonical transformation by

$$L^m := \operatorname{Ad}\psi_< p^m = p^m + \{\Psi_<, p^m\} + \tfrac12\{\Psi_<,\{\Psi_<,p^m\}\} + \tfrac16\{\Psi_<,\{\Psi_<,\{\Psi_<,p^m\}\}\} + \cdots,$$

and compute the first Poisson brackets to obtain

$$\{\Psi_<, p^m\} = -m\Psi_{1,x}p^{m-1} - m\Psi_{2,x}p^{m-2} - m\Psi_{3,x}p^{m-3} + \mathrm{O}(p^{m-4}),$$

$$\begin{aligned}
\{\Psi_<,\{\Psi_<,p^m\}\} &= m(r\Psi_1\Psi_{1,xx} + (m-1)\Psi_{1,x}^2)p^{m-2} + m((r+1)\Psi_2\Psi_{1,xx}\\
&\quad + (2m-3)\Psi_{1,x}\Psi_{2,x} + r\Psi_1\Psi_{2,xx})p^{m-3} + \mathrm{O}(p^{m-4}),
\end{aligned}$$

$$\begin{aligned}
\{\Psi_<,\{\Psi_<,\{\Psi_<,p^m\}\}\} &= -m(r^2\Psi_1^2\Psi_{1,xxx} + r(r+3m-4)\Psi_1\Psi_{1,x}\Psi_{1,xx}\\
&\quad + (m-1)(m-2)\Psi_{1,x}^3)p^{m-3} + \mathrm{O}(p^{m-4})
\end{aligned}$$

when $p\to\infty$. Summing all terms and collecting those with the same power on p we get (24). For $\bar\ell$ we use

$$\bar\ell^m := \operatorname{Ad}\psi_> p^m = p^m + \{\Psi_>, p^m\} + \tfrac12\{\Psi_>,\{\Psi_>,p^m\}\} + \tfrac16\{\Psi_>,\{\Psi_>,\{\Psi_>,p^m\}\}\} + \cdots,$$

and we get equations (25). An alternative way to deduce this is to use the intertwining transformation (15), $(p,x)\to(1/p,-x)$ together $r\to 2-r$, that intertwines $\mathfrak{g}_>$ and $\mathfrak{g}_<$. Thus, the expressions for $\bar v_{m,j}$ are obtained from those for $u_{-m,j}$ by replacing $\partial_x^j\Psi_k$ by $(-1)^j\partial_x^j\bar\Psi_k$ and r by $2-r$.

As $\bar L^m = \operatorname{Ad}_{\psi_{1-r}}\bar\ell^m$ and we have already obtained the expansion of $\bar\ell$ in powers of p, to compute $\bar L^m$, we need to characterize $\operatorname{Ad}_{\exp(\xi p^{1-r})}(\phi(x)p^n)$, i.e. to characterize the canonical transformation generated by ξp^{1-r}. For that aim it is useful to perform the following calculation:

$$\operatorname{ad}_{\xi p^{1-r}}(\phi(x)p^n) = \{\xi p^{1-r}, \phi(x)p^n\} = ([(1-r)\xi\partial_x - n\xi_x](\phi))p^n,$$

so that

$$\operatorname{Ad}_{\exp(\xi p^{1-r})}(\phi(x)p^n) = [\exp((1-r)\xi\partial_x - n\xi_x)\phi(x)]p^n.$$

For $r = 1$, i.e. when $\mathfrak{g}_{1-r}$ is an Abelian Lie subalgebra, the action is easy to compute

$$\mathrm{Ad}_{\exp \xi}(\phi(x)p^n) = \exp(-n\xi_x)\phi(x)p^n.$$

However, for $r \neq 1$, the situation is rather more involved. Let us analyse this non-Abelian situation. The function $\Phi := \exp(\lambda((1-r)\xi\partial_x - n\xi_x)\phi(x)$ is characterized, in a unique manner, by the following initial condition problem for a first-order linear PDE:

$$\partial_\lambda \Phi = (1-r)\xi\partial_x\Phi - n\xi_x\Phi, \qquad \Phi|_{\lambda=0} = \phi,$$

whose solution is

$$\Phi(\lambda, x) = \phi(X(x))\left(\frac{\xi(X(x))}{\xi(x)}\right)^{-n/(1-r)}.$$

Hence

$$\mathrm{Ad}_{\exp(\xi p^{1-r})}(\phi(x)p^n) = \begin{cases} \left(\dfrac{\xi(X(x))}{\xi(x)}\right)^{-n/(1-r)} \phi(X(x))p^n, & \displaystyle\int_x^X \frac{\mathrm{d}x}{\xi(x)} = (1-r), \quad r \neq 1, \\ \exp(-n\xi_x(x))\phi(x)p^n, & r = 1. \end{cases} \tag{26}$$

□

Observe that from the proof of the above proposition, we deduce that $u_{m,j} = -m\Psi_{j+1,x} + U_{m,j}$ where $U_{m,j}$ is a nonlinear function of $\Psi_1, \ldots, \Psi_j$ and its x-derivatives.

3. The associated dispersionless integrable hierarchies

We now deduce the integrable hierarchies associated with the factorization problem (4), namely the rth dispersionless modified KP, rth dispersionless Dym and rth dispersionless Toda hierarchies.

3.1. The rth dispersionless modified KP hierarchy

Now we will study the consequences of (9) and derive a nonlinear PDE for $u_{1,0}$ which resemble the modified dispersionless KP equation, and was found in [22]. For the sake of simplicity we write (9) as

$$\partial_n\psi_< \cdot \psi_<^{-1} + P_< L^{n+1-r} = 0.$$

The right derivatives of $\psi_<$, as follows from (6), are

$$\partial_n\psi_< \cdot \psi_<^{-1} = \partial_n\Psi_< + \tfrac{1}{2}\{\Psi_<, \partial_n\Psi_<\} + \tfrac{1}{6}\{\Psi_<, \{\Psi_<, \partial_n\Psi_<\}\} + \cdots,$$

so that

$$\partial_n\psi_< \cdot \psi_<^{-1} = (\partial_n\Psi_1)p^{-r} + (\partial_n\Psi_2 + \tfrac{1}{2}r(\Psi_{1,x}\partial_n\Psi_1 - \Psi_1\partial_n\Psi_{1,x}))p^{-r-1} + \mathrm{O}(p^{-r-2}) \tag{27}$$

for $p \to \infty$. Recall that when $p \to \infty$ we have

$$L^{n+1-r} = p^{n+1-r} + u_{n+1-r,0}p^{n-r} + u_{n+1-r,1}p^{n-r-1} + u_{n+1-r,2}p^{n-r-2} + \mathrm{O}(p^{n-r-3})$$

with $u_{n+1-r,j} = -(n+1-r)\Psi_{j+1,x} + U_{n+1-r,j}$, and $U_{n+1-r,j}$ a given nonlinear function of $\Psi_1, \ldots, \Psi_j$ and its x-derivatives. Equation (9), together with (27), gives an infinite set of equations, among which the first two are

$$\partial_n \Psi_1 = -(n+1-r)\Psi_{n+1,x} + U_{n+1-r,n},$$

$$\partial_n \Psi_2 + \tfrac{1}{2} r(\Psi_{1,x}\partial_n \Psi_1 - \Psi_1 \partial_n \Psi_{1,x}) = -(n+1-r)\Psi_{n+2,x} + U_{n+1-r,n+1}.$$

Thus, we get for $\Psi_{n+j,x}$, $j = 1, 2, \ldots$, expressions in terms of $\Psi_1, \ldots, \Psi_n$ together with its x-derivatives and integrals and also its ∂_n-derivative. For the next flow we have

$$\partial_{n+1} \Psi_1 = -(n+2-r)\Psi_{n+2,x} + U_{n+2-r,n+1},$$

$$\partial_{n+1} \Psi_2 + \tfrac{1}{2} r(\Psi_{1,x}\partial_{n+1} \Psi_1 - \Psi_1 \partial_{n+1} \Psi_{1,x}) = -(n+2-r)\Psi_{n+3,x} + U_{n+3-r,n+2}$$

from where it follows a nonlinear PDE system for $(\Psi_1, \ldots, \Psi_n)$, in the variables x, t_n, t_{n+1}.

In particular, if $r \neq 2$—when $r = 2$ the t_1-flow is trivial—and $n = 1, 2$ we get

$$\psi_{2,x} = \frac{1}{2-r}\partial_1 \Psi_1 + \frac{1}{2}(r\Psi_1 \Psi_{1,xx} + (1-r)\Psi_{1,x}^2), \tag{28}$$

$$\begin{aligned}\partial_2 \Psi_1 = {} & \frac{3-r}{2-r}\left(\partial_1 \Psi_2 + \frac{r}{2}(\Psi_{1,x}\partial_1 \Psi_1 - \psi_1 \partial_1 \Psi_{1,x})\right) \\ & + (3-r)\left(-\Psi_{1,x}\Psi_{2,x} + \frac{r}{2}\Psi_1 \Psi_{1,x}\Psi_{1,xx} - \frac{r-1}{3}\Psi_{1,x}^3\right)\end{aligned} \tag{29}$$

and, hence,

$$\begin{aligned}\Psi_{1,xt_2} = {} & \frac{3-r}{(2-r)^2}\Psi_{1,t_1t_1} - \frac{(3-r)(1-r)}{2-r}\Psi_{1,xx}\Psi_{1,t_1} \\ & - \frac{(3-r)r}{2-r}\Psi_{1,x}\Psi_{1,xt_1} - \frac{(3-r)(1-r)}{2}\Psi_{1,x}^2\Psi_{1,xx}.\end{aligned} \tag{30}$$

If we introduce

$$u := u_{1,0} = -\Psi_{1,x}, \qquad \partial_x^{-1} u = \int_{x_0}^{x} u(x)\,\mathrm{d}x, \tag{31}$$

we get for u the following nonlinear PDE:

$$u_{t_2} = \frac{3-r}{(2-r)^2}(\partial_x^{-1}u)_{t_1t_1} + \frac{(3-r)(1-r)}{2-r}u_x(\partial_x^{-1}u)_{t_1} + \frac{r(3-r)}{2-r}uu_{t_1} - \frac{(3-r)(1-r)}{2}u^2 u_x. \tag{32}$$

This equation is similar to the dispersionless modified KP equation which is recovered for $r = 0$, hence we called it rth dispersionless modified KP (r-dmKP) equation, note that it was derived for the first time in [22]. Therefore we will refer to equation (30) as the potential r-dmKP equation.

The $\bar{t}_n$ flows for $\psi_>$: From the intertwining property we find out that the $\bar{t}_n$-flows for $\bar{\Psi}_k$ are derived from the t_n-flows for Ψ_k by replacing r by $2-r$, and each $\partial_x^j \Psi_k$ by $(-1)^j \partial_x^j \bar{\Psi}_k$ so that (30) can be written as

$$\begin{aligned}-\bar{\Psi}_{1,x\bar{t}_2} = {} & \frac{1+r}{r^2}\bar{\Psi}_{1,\bar{t}_1\bar{t}_1} + \frac{(1+r)(1-r)}{r}\bar{\Psi}_{1,xx}\bar{\Psi}_{1,\bar{t}_1} \\ & - \frac{(2-r)(1+r)}{r}\bar{\Psi}_{1,x}\bar{\Psi}_{1,x\bar{t}_1} + \frac{(1+r)(1-r)}{2}\bar{\Psi}_{1,x}^2\bar{\Psi}_{1,xx}\end{aligned}$$

or, in terms of $\bar{v} := \bar{v}_1 = -\bar{\Psi}_{1,x}$,

$$-\bar{v}_{\bar{t}_2} = \frac{1+r}{r^2}\partial_x^{-1}\bar{v}_{\bar{t}_1\bar{t}_1} - \frac{(1+r)(1-r)}{r}\bar{v}_x\partial_x^{-1}\bar{v}_{\bar{t}_1} + \frac{(2-r)(1+r)}{r}\bar{v}\bar{v}_{\bar{t}_1} + \frac{(1+r)(1-r)}{2}\bar{v}^2\bar{v}_x.$$

3.2. The rth dispersionless Dym hierarchy

Here we shall discuss the consequences of equations (10) and (13). On the one hand, we may rewrite (10) as

$$P_{1-r}L^{n+1-r} = \partial_n\psi_{1-r}\cdot\psi_{1-r}^{-1},$$

which in terms of $\bar{\psi}_{1-r} := \psi_{1-r}^{-1} = \exp(-\xi p^{1-r})$ reads as

$$\partial_n\bar{\psi}_{1-r}\cdot\bar{\psi}_{1-r}^{-1} + \mathrm{Ad}_{\bar{\psi}_{1-r}}(u_{n+1-r,n-1}p^{1-r}) = 0. \tag{33}$$

On the other, (13) reads as

$$\bar{\partial}_n\psi_{1-r}\cdot\psi_{1-r}^{-1} + \mathrm{Ad}_{\psi_{1-r}}(\bar{v}_{1-r-n,n-1}p^{1-r}) = 0. \tag{34}$$

At this point, it is useful to recall (26) which reads as

$$\mathrm{Ad}_{\exp(\xi p^{1-r})}(f(x)p^n) = \begin{cases} X_x(x)^{-n/(1-r)}f(X(x))p^n, & \displaystyle\int_x^X \frac{\mathrm{d}x}{\xi(x)} = (1-r), \quad r\neq 1,\\ \exp(-n\xi_x(x))f(x)p^n, & r=1,\end{cases}$$

where have used

$$X_x = \frac{\xi(X(x))}{\xi(x)}.$$

In particular, (26) implies, for $r\neq 1$, the following equations:

$$\mathrm{Ad}_{\exp(\xi p^{1-r})}(x) = X, \qquad \int_x^X \frac{\mathrm{d}x}{\xi(x)} = (1-r),$$

$$\mathrm{Ad}_{\exp(-\xi p^{1-r})}(x) = \bar{X}, \qquad \int_x^{\bar{X}} \frac{\mathrm{d}x}{\xi(x)} = -(1-r).$$

Observe that $\bar{X}$ is the inverse function of X, i.e. $\bar{X}\circ X = \mathrm{id}$. This is also a consequence $x = \mathrm{Ad}_{\exp(-\xi p^{1-r})}(X(x)) = \bar{X}(X(x))$. Finally, notice that when $r=1$ we have $X(x) = \bar{X}(x) = x$.

One can show that

$$\partial\psi_{1-r}\cdot\psi_{1-r}^{-1} = \begin{cases} \dfrac{1}{1-r}\dfrac{\partial X}{X_x}p^{1-r}, & \displaystyle\int_x^X \frac{\mathrm{d}x}{\xi(x)} = (1-r), \quad r\neq 0,\\ \partial\xi p^{1-r}, & r=1,\end{cases}$$

$$\partial\bar{\psi}_{1-r}\cdot\bar{\psi}_{1-r}^{-1} = \begin{cases} \dfrac{1}{1-r}\dfrac{\partial\bar{X}}{\bar{X}_x}p^{1-r}, & \displaystyle\int_x^{\bar{X}} \frac{\mathrm{d}x}{\xi(x)} = -(1-r), \quad r\neq 0,\\ -\partial\xi p^{1-r}, & r=1.\end{cases}$$

We are now ready to tackle (33) and (34), which read

$$\begin{cases}\partial_n \bar{X} = -(1-r)u_{n+1-r,n-1}(\bar{X}), & r \neq 1,\\ \partial_n \xi = u_{n,n-1}, & r = 1,\end{cases} \tag{35}$$

$$\begin{cases}\bar{\partial}_n X = -(1-r)\bar{v}_{1-r-n,n-1}(X), & r \neq 1,\\ \bar{\partial}_n \xi = -\bar{v}_{-n,n-1}, & r = 1.\end{cases} \tag{36}$$

Let us look at the consequences of (35) for $n = 1$ and $n = 2$ and recall (24):

$$\begin{cases}\bar{X}_{t_1} = (1-r)(2-r)\Psi_{1,x}(\bar{X}), & r \neq 1,\\ \xi_{t_1} = -\Psi_{1,x}, & r = 1,\end{cases} \tag{37}$$

$$\begin{cases}\bar{X}_{t_2} = -(1-r)(3-r)(\Psi_{2,x}(\bar{X}) & \\ \qquad -\frac{1}{2}(r\Psi_1(\bar{X})\Psi_{1,xx}(\bar{X}) + (2-r)\Psi_{1,x}(\bar{X})^2)), & r \neq 1,\\ \xi_{t_2} = -2(\Psi_{2,x} - \frac{1}{2}(\Psi_1\Psi_{1,xx} + \Psi_{1,x}^2)), & r = 1.\end{cases} \tag{38}$$

We first analyse the case $r \neq 1$. The first equation of (37), when $r \neq 1, 2$, gives the important relation

$$\Psi_{1,x}(\bar{X}) = \frac{1}{(1-r)(2-r)}\bar{X}_{t_1}. \tag{39}$$

By introducing (28) into (38) we get

$$\bar{X}_{t_2} = (1-r)(3-r)\left(\frac{1}{2-r}\Psi_{1,t_1}(\bar{X}) - \frac{1}{2}\Psi_{1,x}(\bar{X})^2\right),$$

which we will manipulate. Firstly, we take its x-derivative

$$\bar{X}_{xt_2} = (1-r)(3-r)\left(\frac{1}{2-r}\Psi_{1,t_1x}(\bar{X})\bar{X}_x - \Psi_{1,x}(\bar{X})(\Psi_{1,x}(\bar{X}))_x\right);$$

secondly we see that

$$\Psi_{1,t_1x}(\bar{X})\bar{X}_x = ((\Psi_{1,x}(\bar{X}))_{t_1} - \Psi_{1,xx}(\bar{X})\bar{X}_{t_1})\bar{X}_x = (\Psi_{1,x}(\bar{X}))_{t_1}\bar{X}_x - (\Psi_{1,x}(\bar{X}))_x\bar{X}_{t_1}.$$

Therefore,

$$\bar{X}_{xt_2} = (1-r)(3-r)\left(\frac{1}{2-r}((\Psi_{1,x}(\bar{X}))_{t_1}\bar{X}_x - (\Psi_{1,x}(\bar{X}))_x\bar{X}_{t_1}) - \Psi_{1,x}(\bar{X})(\Psi_{1,x}(\bar{X}))_x\right),$$

and recalling (39) we have

$$\bar{X}_{xt_2} = \frac{3-r}{2-r}\left(\frac{1}{2-r}\bar{X}_{t_1t_1}\bar{X}_x - \frac{1}{1-r}\bar{X}_{xt_1}\bar{X}_{t_1}\right). \tag{40}$$

For $r = 1$ we introduce (28) into (38) to get

$$\xi_{t_2} = -2\Psi_{1,t_1} + \Psi_{1,x}^2,$$

and taking the x-derivative we obtain

$$\xi_{xt_2} = -2\Psi_{1,xt_1} + 2\Psi_{1,x}\Psi_{1,xx},$$

and recalling (36) for $r = 1$ is

$$\xi_{t_2x} - 2\xi_{t_1}\xi_{t_1x} - 2\xi_{t_1t_1} = 0. \tag{41}$$

Upon the introduction of the variable

$$v = \begin{cases} (\bar{X}_x)^{-1/(1-r)}, & r \neq 1, \\ \exp \xi_x, & r = 1, \end{cases} \tag{42}$$

equations (38) and (39) transform onto

$$v_{t_2} = \frac{3-r}{(2-r)^2} v^{r-1}(v^{2-r}\partial_x^{-1}(v^{r-2}v_{t_1}))_{t_1} \tag{43}$$

which resembles the dispersionless Dym equation which appears for $r = 0$, hence we refer to it as the rth dispersionless Dym (r-dDym) equation (for $r \neq 2$); this equation was first derived in [22]. We shall refer to (38) as the potential r-dDym equation.

From (36) we derive

$$\begin{cases} X_{\bar{t}_1} = & -(1-r)r\bar{\Psi}_{1,x}(X), \quad r \neq 1, \\ \xi_{\bar{t}_1} = & -\bar{\Psi}_{1,x}, \quad r = 1, \end{cases} \tag{44}$$

$$\begin{cases} X_{\bar{t}_2} = -(1-r)(1+r)(\bar{\Psi}_{2,x}(X) \\ \qquad + \frac{1}{2}((2-r)\bar{\Psi}_1(X)\bar{\Psi}_{1,xx}(X) + r\bar{\Psi}_{1,x}(X)^2)), & r \neq 1, \\ \xi_{\bar{t}_2} = -2(\bar{\Psi}_{2,x} + \frac{1}{2}(\bar{\Psi}_1\bar{\Psi}_{1,xx} + \bar{\Psi}_{1,x}^2)), & r = 1, \end{cases} \tag{45}$$

which, for $r \neq 1$, lead to

$$-X_{x\bar{t}_2} = \frac{1+r}{r}\left(\frac{1}{r}X_{\bar{t}_1\bar{t}_1}X_x + \frac{1}{1-r}X_{x\bar{t}_1}X_{\bar{t}_1}\right), \tag{46}$$

and when $r = 1$ to

$$\xi_{\bar{t}_2x} + 2\xi_{\bar{t}_1}\xi_{\bar{t}_1x} + 2\xi_{\bar{t}_1\bar{t}_1} = 0. \tag{47}$$

3.3. Miura map among r-dmKP and r-dDym equations

From (31), (37) and (40) we get

$$\begin{cases} -\frac{1}{(1-r)(2-r)}\bar{X}_{t_1} = u(\bar{X}), \quad \bar{X} = \partial_x^{-1}v^{r-1}, & r \neq 1, \\ \xi_{t_1} = u, \quad \xi = \partial_x^{-1}\log v, & r = 1 \end{cases}$$

and

$$\begin{cases} \partial_x^{-1}(v^{r-1})_{t_1} = -(2-r)(1-r)u(\partial_x^{-1}v^{r-1}), & r \neq 1, \\ \partial_x^{-1}(\log v)_{t_1} = u(x), & r = 1. \end{cases} \tag{48}$$

Equations (48) relate solutions u and v of the r-dmKP (32) and r-dDym (43) equations.

If we derive the above relations with respect to x we get

$$\begin{cases} u_x(\partial_x^{-1}v^{r-1}) = \dfrac{1}{2-r}(\log v)_{t_1}, & r \neq 1, \\ u_x = (\log v)_{t_1}, & r = 1. \end{cases} \tag{49}$$

In any case, observe that a solution v to the r-dDym equation provides us with a solution of the r-dmKP equation, after the calculation of some inverse functions of $\bar{X}$, but the reverse, given a solution u of the r-dmKP equation (32) to get v a solution of the r-dDym equation (43) do not follow from either from (48) or (49). Observe that, in [26], a similar Miura map was derived, in a quite different manner, for the well known $r = 0$ case.

3.4. The rth dispersionless Toda hierarchy

Consider equations (11) and (12)

$$\mathrm{Ad}_{\bar{\psi}_{1-r}} P_> L^{n+1-r} = \partial_n \psi_> \cdot \psi_>^{-1}, \qquad \mathrm{Ad}_{\psi_{1-r}} P_< \bar{\ell}^{1-r-n} = \bar{\partial}_n \psi_< \cdot \psi_<^{-1},$$

which, for $n = 1$, reads

$$\mathrm{Ad}_{\bar{\psi}_{1-r}} P_> L^{2-r} = \partial_1 \psi_> \cdot \psi_>^{-1}, \tag{50}$$

$$\mathrm{Ad}_{\psi_{1-r}} P_< \bar{\ell}^{-r} = \bar{\partial}_1 \psi_< \cdot \psi_<^{-1}. \tag{51}$$

Looking at the leading terms in p we obtain from (50) and (51) the following equations:

$$\bar{\Psi}_{1,t_1} = \begin{cases} (\bar{X}_x)^{-(2-r)/(1-r)}, & r \neq 1, \\ \exp(\xi_x), & r = 1, \end{cases}$$

$$\Psi_{1,\bar{t}_1} = \begin{cases} (X_x)^{r/(1-r)}, & r \neq 1, \\ \exp(\xi_x), & r = 1. \end{cases}$$

Recall now equations (39) and (44) which, taking into account $\bar{X}_{t_1}(X) = -X_{t_1}(x)/X_x(x)$ and $X_{\bar{t}_1}(\bar{X}) = -\bar{X}_{\bar{t}_1}(x)/\bar{X}_x(x)$, can be written as

$$\Psi_{1,x} = \begin{cases} -\dfrac{1}{(1-r)(2-r)} \dfrac{X_{t_1}}{X_x}, & r \neq 1, \\ -\xi_{t_1}, & r = 1, \end{cases}$$

$$\bar{\Psi}_{1,x} = \begin{cases} \dfrac{1}{(1-r)r} \dfrac{\bar{X}_{\bar{t}_1}}{\bar{X}_x}, & r \neq 1, \\ -\xi_{\bar{t}_1}, & r = 1, \end{cases}$$

respectively. The compatibility of these equations lead to

$$((\bar{X}_x)^{-(2-r)/(1-r)})_x - \frac{1}{(1-r)r}\left(\frac{\bar{X}_{\bar{t}_1}}{\bar{X}_x}\right)_{t_1} = 0, \tag{52a}$$

$$((X_x)^{(r/(1-r)})_x + \frac{1}{(1-r)(2-r)}\left(\frac{X_{t_1}}{X_x}\right)_{\bar{t}_1} = 0, \tag{52b}$$

when $r \neq 1$, while for $r = 1$ the equation is

$$(\exp(\xi_x))_x + \xi_{t_1\bar{t}_1} = 0; \tag{53}$$

these are new integrable equations, which we call rth dispersionles Toda (r-dToda) equation, because for $r = 1$ the corresponding equation is the dispersionless Toda equation—known also as the Boyer–Finley equation.

Equations (52a) and (52b) are the same equation, indeed. To prove it we just need to evaluate equation (52b) on $\bar{X}$ and recall that X is the inverse function of $\bar{X}$, $X(\bar{X}(x)) = x$.

4. Additional or master symmetries

In this section, we deal with the additional or master symmetries of the integrable hierarchies just described. We first introduced the Orlov functions $M, \bar{M}, \bar{m}$ in this context and consider the construction of additional symmetries. We compute explicitly some of these additional symmetries for the potential r-dmKP (30), the r-dDym (40) and the r-dToda (52a) equations, finding explicit symmetries of these nonlinear equations depending on arbitrary functions of the variable t_2.

4.1. The Orlov funtions

In formulae (17) we introduced the Lax functions $L, \bar{\ell}$ and $\bar{L}$, which are the canonical transformation of the p variable through $\psi_<, \psi_>$ and $\psi_\geqslant$, respectively. Recalling that t and $\bar{t}$ are functions of p only we can write these Lax functions as follows:

$$L = \mathrm{Ad}_{\psi_<\cdot\exp t}\, p, \qquad \bar{\ell} = \mathrm{Ad}_{\psi_>\cdot\exp\bar{t}}\, p, \qquad \bar{L} = \mathrm{Ad}_{\psi_\geqslant\cdot\exp\bar{t}}\, p.$$

The Orlov functions $M, \bar{m}$ and $\bar{M}$ are defined analogously with the replacement of p by x:

$$M := \mathrm{Ad}_{\psi_<\cdot\exp t}\, x, \qquad \bar{m} := \mathrm{Ad}_{\psi_>\cdot\exp\bar{t}}\, x, \qquad \bar{M} := \mathrm{Ad}_{\psi_\geqslant\cdot\exp\bar{t}}\, x. \tag{54}$$

In the next proposition we describe the form of the Orlov functions as series in the Lax functions.

Proposition 2. *The Orlov functions defined in* (54) *have the following expansions:*

$$M = \cdots + w_2 L^{-2} + w_1 L^{-1} + x + (2-r)t_1 L + (3-r)t_2 L^2 + \cdots, \quad L \to \infty,$$

$$\bar{m} = \cdots - (r+1)\bar{t}_2\bar{\ell}^{-2} - r\bar{t}_1\bar{\ell}^{-1} + x + \bar{\omega}_1\bar{\ell} + \bar{\omega}_2\bar{\ell}^2 + \cdots, \quad \bar{\ell} \to 0,$$

$$\bar{M} = \cdots - (r+1)\bar{t}_2\bar{L}^{-2} - r\bar{t}_1\bar{L}^{-1} + X + \bar{w}_1\bar{L} + \bar{w}_2\bar{L}^2 + \cdots, \quad \bar{L} \to 0,$$

where, for example,

$$w_1 = -r\Psi_1, \qquad w_2 = -(r+1)(\Psi_2 - \tfrac{1}{2}r\Psi_1\Psi_{1,x}),$$
$$\bar{\omega}_1 = (2-r)\bar{\Psi}_1(x), \qquad \bar{\omega}_2 = (3-r)(\bar{\Psi}_2(x) - \tfrac{1}{2}(2-r)\bar{\Psi}_1(x)\bar{\Psi}_{1,x}(x)),$$
$$\bar{w}_1 = (2-r)\bar{\Psi}_1(X), \qquad \bar{w}_2 = (3-r)(\bar{\Psi}_2(X) - \tfrac{1}{2}(2-r)\bar{\Psi}_1(X)\bar{\Psi}_{1,x}(X)).$$

Proof. Now, taking into account that

$$\mathrm{ad}_t\, x = \{t, x\} = p^r \frac{\partial t}{\partial p} = (2-r)t_1 p + (3-r)t_2 p^2 + \cdots$$

we evaluate

$$\mathrm{Ad}_{\exp t} x = \exp(\mathrm{ad}_t)(x) = x + p^r \frac{\partial t}{\partial p} = x + (2-r)t_1 p + (3-r)t_2 p^2 + \cdots,$$

and, therefore,

$$\begin{aligned} M &= \mathrm{Ad}_{\psi_<}\left(x + p^r \frac{\partial t}{\partial p}\right) = \mathrm{Ad}_{\psi_<}(x) + L^r \frac{\partial t(L)}{\partial L} \\ &= \mathrm{Ad}_{\psi_<}(x) + (2-r)t_1 L + (3-r)t_2 L^2 + \cdots. \end{aligned}$$

To compute M we need to evaluate

$$\mathrm{ad}_{\Psi_<}(x) = \{\Psi_<, x\} = p^r \frac{\partial \Psi_<}{\partial p} = D_p \Psi_<,$$

where

$$D_p := p^r \frac{\partial}{\partial p}.$$

Notice that D_p is a derivation of the Lie algebra $\mathfrak{g}$:

$$D_p\{f, g\} = \{D_p f, g\} + \{f, D_p g\}.$$

Thus

$$\mathrm{Ad}_{\psi_<}(x) = x + \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \mathrm{ad}^n_{\Psi_<} D_p \Psi_< = x + D_p \psi_< \cdot \psi_<^{-1},$$

where

$$D_p \Psi_< = -(r\Psi_1 p^{-1} + (r+1)\Psi_2 p^{-2} + \cdots).$$

We can compute now $D_p \psi_< \cdot \psi_<^{-1}$:

$$\begin{aligned} D_p \psi_< \cdot \psi_<^{-1} &= D_p \Psi_< + \tfrac{1}{2}\{\Psi_<, D_p \Psi_<\} + \cdots \\ &= -r\Psi_1 p^{-1} - ((r+1)\Psi_2 - \tfrac{1}{2}r(r-1)\Psi_1\Psi_{1,x})p^{-2} + \cdots. \end{aligned}$$

From

$$L^m = p^m + u_{m,0} p^{m-1} + u_{m,1} p^{m-2} + \cdots,$$

we deduce

$$p^m = L^m - u_{m,0} L^{m-1} - (u_{m,1} - u_{m-1,1} u_{m,0}) L^{m-2} - \cdots$$

so that

$$M = \cdots + w_2 L^{-2} + w_1 L^{-1} + x + (2-r)t_1 L + (3-r)t_2 L^2 + \cdots, \qquad (55)$$

where, for example,

$$w_1 = -r\Psi_1, \qquad w_2 = -(r+1)(\Psi_2 - \tfrac{1}{2}r\Psi_1\Psi_{1,x}).$$

For $\bar{m}$ and $\bar{M}$ we proceed in a similar manner. Firstly

$$\begin{aligned}
\mathrm{ad}_{\bar{t}}\, x &= \{\bar{t}, x\} = p^r \frac{\partial \bar{t}}{\partial p} = -r\bar{t}_1 p^{-1} - (r+1)\bar{t}_2 p^{-2} + \cdots \\
\mathrm{Ad}_{\exp \bar{t}}\, x &= x + p^r \frac{\partial \bar{t}}{\partial p} = x - r\bar{t}_1 p^{-1} - (r+1)\bar{t}_2 p^{-2} + \cdots,
\end{aligned}$$

so that

$$\begin{aligned}
\bar{m} &= \mathrm{Ad}_{\psi_>}(x) + \bar{\ell}^r \frac{\partial \bar{t}(\bar{\ell})}{\partial \bar{\ell}} = \mathrm{Ad}_{\psi_>}(x) - r\bar{t}_1\bar{\ell}^{-1} - (r+1)\bar{t}_2\bar{\ell}^{-2} + \cdots, \\
\bar{M} &= \mathrm{Ad}_{\psi_\geqslant}(x) + \bar{L}^r \frac{\partial t(\bar{L})}{\partial \bar{L}} = \mathrm{Ad}_{\psi_\geqslant}(x) - r\bar{t}_1\bar{L}^{-1} - (r+1)\bar{t}_2\bar{L}^{-2} + \cdots.
\end{aligned}$$

Now

$$\mathrm{Ad}_{\psi_>}(x) = x + D_p\psi_> \cdot \psi_>^{-1}$$

and

$$D_p\Psi_> = (2-r)\bar{\Psi}_1 p + (3-r)\bar{\Psi}_2 p^2 + \cdots.$$

Hence,

$$\begin{aligned}
D_p\psi_> \cdot \psi_>^{-1} &= D_p\bar{\Psi} + \tfrac{1}{2}\{\bar{\Psi}, D_p\bar{\Psi}\} + \cdots \\
&= (2-r)\bar{\Psi}_1 p + ((3-r)\bar{\Psi}_2 + \tfrac{1}{2}(2-r)(1-r)\bar{\Psi}_1\bar{\Psi}_{1,x})p^2 + \cdots.
\end{aligned}$$

From

$$\bar{\ell}^m = p^m + \bar{v}_{m,0}p^{m+1} + \bar{v}_{m,1}p^{m+2} + \cdots,$$

we deduce

$$p^m = \bar{\ell}^m - \bar{v}_{m,0}\bar{\ell}^{m+1} - (\bar{v}_{m,1} - \bar{v}_{m+1,0}\bar{v}_{m,0})\bar{\ell}^{m-2} - \cdots$$

so that

$$\begin{aligned}
\bar{m} = {} & \cdots + (3-r)(\bar{\Psi}_2 - \tfrac{1}{2}(2-r)\bar{\Psi}_1\bar{\Psi}_{1,x})\bar{\ell}^2 \\
& + (2-r)\bar{\Psi}_1(x)\bar{\ell} + x - r\bar{t}_1\bar{\ell}^{-1} - (r+1)\bar{t}_2\bar{\ell}^{-2} + \cdots.
\end{aligned}$$

Therefore,

$$\bar{M} = \mathrm{Ad}_{\psi_{1-r}}\bar{m} = \cdots - (r+1)\bar{t}_2\bar{L}^{-2} - r\bar{t}_1\bar{L}^{-1} + X + \bar{w}_1\bar{L} + \bar{w}_2\bar{L}^2 + \cdots, \tag{56}$$

with

$$\bar{w}_1 = (2-r)\bar{\Psi}_1(X), \qquad \bar{w}_2 = (3-r)(\bar{\Psi}_2(X) - \tfrac{1}{2}(2-r)\bar{\Psi}_1(X)\bar{\Psi}_{1,x}(X)). \qquad \square$$

We now find the Lax equations for M and $\bar{M}$. For that aim we compute

$$\partial_n(\psi_< \cdot \exp t) \cdot (\psi_< \cdot \exp t)^{-1} = \partial\psi_< \cdot \psi_<^{-1} + \mathrm{Ad}_{\psi_<}(p^{n+1-r}) = P_\geqslant L^{n+1-r},$$

$$\bar{\partial}_n(\psi_\geqslant \cdot \exp \bar{t}) \cdot (\psi_\geqslant \cdot \exp \bar{t})^{-1} = \partial\psi_\geqslant \cdot \psi_\geqslant^{-1} + \mathrm{Ad}_{\psi_\geqslant}(p^{-n+1-r}) = P_< \bar{L}^{-n+1-r}$$

and conclude

Proposition 3. *The Lax and Orlov functions are subject to the following Lax equations:*

$$\begin{cases} \partial_n L = \{P_\geqslant L^{n+1-r}, L\}, & \partial_n \bar{L} = \{P_\geqslant L^{n+1-r}, \bar{L}\}, \\ \bar{\partial}_n L = \{P_< \bar{L}^{1-r-n}, L\}, & \bar{\partial}_n \bar{L} = \{P_< \bar{L}^{1-r-n}, \bar{L}\}, \end{cases}$$

$$\begin{cases} \partial_n M = \{P_\geqslant L^{n+1-r}, M\}, & \partial_n \bar{M} = \{P_\geqslant L^{n+1-r}, \bar{M}\}, \\ \bar{\partial}_n M = \{P_< \bar{L}^{1-r-n}, M\}, & \bar{\partial}_n \bar{M} = \{P_< \bar{L}^{1-r-n}, \bar{M}\}. \end{cases}$$

4.2. Additional or master symmetries and its generators

Let us consider that the initial conditions $h, \bar{h}$ in the factorization problem describe smooth curves $h(s) = \exp(H(s)), \bar{h}(s) = \exp(\bar{H}(s))$ in G—here $H(s)$ and $\bar{H}(s)$ are curves in $\mathfrak{g}$. This implies that the factors $\psi_< = \psi_<(s)$ and $\psi_\geqslant = \psi_\geqslant(s)$ in the corresponding factorization problem do depend on s:

$$\psi_<(s) \cdot \exp(t) \cdot h(s) = \psi_\geqslant(s) \cdot \exp(\bar{t}) \cdot \bar{h}(s). \tag{57}$$

If we introduce the notation

$$F := \frac{\mathrm{d}h}{\mathrm{d}s} \cdot h^{-1}, \qquad \bar{F} := \frac{\mathrm{d}\bar{h}}{\mathrm{d}s} \cdot \bar{h}^{-1} \tag{58}$$

and take the right derivative with respect to s of (57) we get

$$\frac{\mathrm{d}\psi_<}{\mathrm{d}s} \cdot \psi_<^{-1} + \mathrm{Ad}_{\psi_< \cdot \exp t}(F(p,x)) = \frac{\mathrm{d}\psi_\geqslant}{\mathrm{d}s} \cdot \psi_\geqslant^{-1} + \mathrm{Ad}_{\psi_\geqslant \cdot \exp \bar{t}}(\bar{F}(p,x))$$

that implies

$$\frac{\mathrm{d}\psi_\geqslant}{\mathrm{d}s} \cdot \psi_\geqslant^{-1} - \frac{\mathrm{d}\psi_<}{\mathrm{d}s} \cdot \psi_<^{-1} = F(L,M) - \bar{F}(\bar{L},\bar{M}).$$

Now, we may split this equation into

$$\begin{aligned} \frac{\mathrm{d}\psi_<}{\mathrm{d}s} \cdot \psi_<^{-1} &= -P_<(F(L,M) - \bar{F}(\bar{L},\bar{M})), \\ \frac{\mathrm{d}\psi_\geqslant}{\mathrm{d}s} \cdot \psi_\geqslant^{-1} &= P_\geqslant(F(L,M) - \bar{F}(\bar{L},\bar{M})). \end{aligned} \tag{59}$$

Equations (59) imply for the Lax and Orlov functions $L, \bar{L}, M$ and $\bar{M}$ the

Proposition 4. *The Lax and Orlov functions are transformed by the additional symmetries according to the following formulae:*

$$\begin{aligned} \frac{\mathrm{d}L}{\mathrm{d}s} &= \{-P_<(F(L,M) - \bar{F}(\bar{L},\bar{M})), L\}, & \frac{\mathrm{d}M}{\mathrm{d}s} &= \{-P_<(F(L,M) - \bar{F}(\bar{L},\bar{M})), M\}, \\ \frac{\mathrm{d}\bar{L}}{\mathrm{d}s} &= \{P_\geqslant(F(L,M) - \bar{F}(\bar{L},\bar{M})), \bar{L}\}, & \frac{\mathrm{d}\bar{M}}{\mathrm{d}s} &= \{P_\geqslant(F(L,M) - \bar{F}(\bar{L},\bar{M})), \bar{M}\}. \end{aligned} \tag{60}$$

Without loss of generality, if we take $\bar{h}(s) = \mathrm{id}$, then $\bar{F} = 0$ and (59) read as

$$\frac{\mathrm{d}\psi_<}{\mathrm{d}s} \cdot \psi_<^{-1} = -P_<(F(L, M)), \tag{61}$$

$$\frac{\mathrm{d}\psi_\geqslant}{\mathrm{d}s} \cdot \psi_\geqslant^{-1} = P_\geqslant(F(L, M)). \tag{62}$$

Alternatively, we could set $h(s) = \mathrm{id}$ so that

$$\frac{\mathrm{d}\psi_<}{\mathrm{d}s} \cdot \psi_<^{-1} = P_<(\bar{F}(\bar{L}, \bar{M})), \tag{63}$$

$$\frac{\mathrm{d}\psi_\geqslant}{\mathrm{d}s} \cdot \psi_\geqslant^{-1} = -P_\geqslant(\bar{F}(\bar{L}, \bar{M})). \tag{64}$$

Hereafter we shall assume that

$$F(L, M) = \sum c_{ij} L^i M^j, \qquad \bar{F}(\bar{L}, \bar{M}) = \sum \bar{c}_{ij} \bar{L}^i \bar{M}^j. \tag{65}$$

Let us keep $t_n = 0$ for $n > N$ and $\bar{t}_n = 0$ for $n > \bar{N}$; then recalling (55) we have

$$M = (N + 1 - r)t_N L^N + \cdots + (2 - r)t_1 L + x + w_1 L^{-1} + w_2 L^{-2} + \cdots, \tag{66}$$

$$\bar{M} = -(r + \bar{N} - 1)\bar{t}_{\bar{N}} \bar{L}^{-\bar{N}} - \cdots - r\bar{t}_1 \bar{L}^{-1} + X + \bar{w}_1 \bar{L} + \bar{w}_2 \bar{L}^2 + \cdots. \tag{67}$$

Notice that if we want to keep $t_n = 0$ for $n > N$ within the transformation given by the symmetry, then (18) imply that the function $F(L, M)$, when M is expressed as in (66), has no terms proportional to L^n for $n > N - r + 1$. But as F has the form indicated in (65) we only need to impose this condition over each of the products $L^i M^j$:

$$\begin{aligned} L^i M^j &= L^i((N + 1 - r)t_N L^N + \cdots + (2 - r)t_1 L + x + w_1 L^{-1} + w_2 L^{-2} + \cdots)^j \\ &= ((N + 1 - r)t_N)^j L^{i+Nj} + \cdots \Rightarrow c_{ij} = 0 \text{ if } i + Nj > N - r + 1. \end{aligned}$$

Hence,

$$F(L, M) = \sum_{n=1-r}^{N-r+1} \alpha_n \left(\frac{M}{(N + 1 - r)L^N}\right) L^n \tag{68}$$

with α_n analytic functions.

The same reasoning may be applied to keep $\bar{t}_n = 0$ for $n > \bar{N}$, and the corresponding condition is

$$\bar{F}(\bar{L}, \bar{M}) = \sum_{n=1-r}^{\bar{r}-\bar{N}+1} \bar{\alpha}_n \left(\frac{\bar{M}}{(\bar{N} - 1 + r)\bar{L}^{-\bar{N}}}\right) \bar{L}^{-n} \tag{69}$$

with $\bar{\alpha}_n$ analytic functions.

4.3. *Symmetries of the potential r-dmKP equation*

In the following lines we shall find three symmetries of the r-dmKP equation (30). Let us suppose that $N = 2$, and $n = 1-r$, $2-r$ and $3-r$, so that we have three different contributions, or generators, to F, namely

$$\alpha\left(\frac{M}{(3-r)L^2}\right)L^{1-r}, \qquad \alpha\left(\frac{M}{(3-r)L^2}\right)L^{2-r} \qquad \text{and} \qquad \alpha\left(\frac{M}{(3-r)L^2}\right)L^{3-r}.$$

We first observe that

$$\frac{M}{(3-r)L^2}=t_2+\frac{2-r}{3-r}t_1L^{-1}+\frac{1}{3-r}xL^{-2}-\frac{r}{3-r}\Psi_1L^{-3}+\cdots.$$

If we denote

$$\varepsilon:=\frac{2-r}{3-r}t_1L^{-1}+\frac{1}{3-r}xL^{-2}-\frac{r}{3-r}\Psi_1L^{-3}+\cdots$$

we have the following Taylor expansion:

$$\begin{aligned}\alpha(t_2+\varepsilon)&=\alpha(t_2)+\dot\alpha(t_2)\varepsilon+\tfrac{1}{2}\ddot\alpha(t_2)\varepsilon^2+\tfrac{1}{6}\dddot\alpha(t_2)\varepsilon^3+\cdots\\&=\alpha(t_2)+\frac{2-r}{3-r}\dot\alpha(t_2)t_1L^{-1}+\left(\frac{1}{3-r}\dot\alpha(t_2)x+\frac{(2-r)^2}{2(3-r)^2}\ddot\alpha(t_2)t_1^2\right)L^{-2}\\&\quad+\left(-\frac{r}{3-r}\dot\alpha(t_2)\Psi_1+\frac{2-r}{(3-r)^2}\ddot\alpha(t_2)t_1x+\frac{(2-r)^3}{6(3-r)^3}\dddot\alpha(t_2)t_1^3\right)L^{-3}+\cdots.\end{aligned}$$

We shall now analyse each of the three cases:

1. Now, we have

$$F=\alpha(t_2+\varepsilon)L^{1-r}=\alpha(t_2)L^{1-r}+\frac{2-r}{3-r}\dot\alpha(t_2)t_1L^{-r}+\cdots,$$

so that

$$\frac{\mathrm{d}\psi_<}{\mathrm{d}s}\cdot\psi_<^{-1}=-P_<F=-\alpha(t_2)P_<(L^{1-r})-\frac{2-r}{3-r}\dot\alpha(t_2)t_1L^{-r}+\cdots.$$

Recalling that

$$\frac{\mathrm{d}\psi_<}{\mathrm{d}s}\cdot\psi_<^{-1}=(\partial_s\Psi_1)p^{-r}+\left(\partial_s\Psi_2+\frac{r}{2}(\Psi_{1,x}\partial_s\Psi_1-\Psi_1\partial_s\Psi_{1,x})\right)p^{-r-1}+\cdots,$$

we deduce for Ψ_1 the following PDE:

$$\Psi_{1,s}=(1-r)\alpha(t_2)\Psi_{1,x}-\frac{2-r}{3-r}\dot\alpha(t_2)t_1$$

whose solution is given by

$$\Psi_1(s)=\frac{2-r}{(1-r)(3-r)}\frac{\dot\alpha(t_2)}{\alpha(t_2)}t_1x+f\left(t_1,t_2,s+\frac{x}{(1-r)\alpha(t_2)}\right)$$

with f and arbitrary function.

For $s=0$ we obtain

$$\Psi_1=\Psi_1(s)|_{s=0}=\frac{2-r}{(1-r)(3-r)}\frac{\dot\alpha(t_2)}{\alpha(t_2)}t_1x+f\left(t_1,t_2,\frac{x}{(1-r)\alpha(t_2)}\right)$$

from which we obtain

$$\begin{aligned}\Psi_1(x+(1-r)s\alpha(t_2),t_1,t_2)&=\frac{2-r}{(1-r)(3-r)}\frac{\dot\alpha(t_2)}{\alpha(t_2)}t_1(x+(1-r)s\alpha(t_2))\\&\quad+f\left(t_1,t_2,s+\frac{x}{(1-r)\alpha(t_2)}\right).\end{aligned}$$

Hence,

$$\Psi_1(s) = -\frac{2-r}{3-r} s\dot{\alpha}(t_2)t_1 + \Psi_1(x + (1-r)s\alpha(t_2), t_1, t_2).$$

Therefore, we conclude that given any solution $\Psi_1(x, t_1, t_2)$ of the potential r-dmKP equation (30) and any analytic function $\alpha(t_2)$ then

$$\tilde{\Psi}_1(x, t_1, t_2) := -\frac{2-r}{3-r}\dot{\alpha}(t_2)t_1 + \Psi_1(x + (1-r)\alpha(t_2), t_1, t_2)$$

is a new solution of the equation. For $u = -\Psi_{1,x}$ the symmetry transformation for a solution of the r-dmKP equation (32) is given by

$$\tilde{u}(x, t_1, t_2) = u(x + (1-r)\alpha(t_2), t_1, t_2),$$

which for $r = 1$ is the identity transformation.

2. In this case

$$\begin{aligned} F &= \alpha(t_2 + \varepsilon)L^{2-r} \\ &= \alpha(t_2)L^{2-r} + \frac{2-r}{3-r}\dot{\alpha}(t_2)t_1L^{1-r} + \left(\frac{1}{3-r}\dot{\alpha}(t_2)x + \frac{(2-r)^2}{2(3-r)^2}\ddot{\alpha}(t_2)t_1^2\right)L^{-r} + \cdots \end{aligned}$$

so that

$$\begin{aligned} \frac{\mathrm{d}\psi_<}{\mathrm{d}s}\cdot\psi_<^{-1} = \alpha(t_2)\frac{\mathrm{d}\psi_<}{\mathrm{d}t_1}\cdot\psi_<^{-1} - \frac{2-r}{3-r}\dot{\alpha}(t_2)t_1P_<L^{1-r} \\ - \left(\frac{1}{3-r}\dot{\alpha}(t_2)x + \frac{(2-r)^2}{2(3-r)^2}\ddot{\alpha}(t_2)t_1^2\right)L^{-r} + \cdots. \end{aligned}$$

For Ψ_1 we find the following PDE:

$$\Psi_{1,s} = \alpha(t_2)\Psi_{1,t_1} + \frac{(1-r)(2-r)}{3-r}\dot{\alpha}(t_2)t_1\Psi_{1,x} - \frac{1}{3-r}\dot{\alpha}(t_2)x - \frac{(2-r)^2}{2(3-r)^2}\ddot{\alpha}(t_2)t_1^2$$

whose solution is given by

$$\Psi_1(s) = g(x, t_1, t_2) + f\left(t_2, t_1^2 - \frac{2(3-r)}{(1-r)(2-r)}\frac{\alpha}{\dot{\alpha}}x, s + \frac{t_1}{\alpha}\right) \tag{70}$$

with f being an arbitrary function and

$$g := \frac{1}{3-r}\frac{\dot{\alpha}}{\alpha}xt_1 + \frac{(2-r)}{6(3-r)^2}\left((2-r)\frac{\ddot{\alpha}}{\alpha} - 2(1-r)\frac{\dot{\alpha}^2}{\alpha^2}\right)t_1^3.$$

Setting $s = 0$ in (70) we arrive at

$$\Psi_1 = \Psi_1\big|_{s=0} = g(x, t_1, t_2) + f\left(t_2, t_1^2 - \frac{2(3-r)}{(1-r)(2-r)}\frac{\alpha}{\dot{\alpha}}x, \frac{t_1}{\alpha}\right). \tag{71}$$

If in (71) we replace the independent variables x and t_1 by

$$x(s) = x + s\left(\frac{(1-r)(2-r)}{2(3-r)}\dot{\alpha}(s\alpha + 2t_1)\right), \qquad \tilde{t}_1(s) = t_1 + s\alpha,$$

we deduce

$$f\left(t_2, t_1^2 - \frac{2(3-r)}{(1-r)(2-r)}\frac{\alpha}{\dot{\alpha}}x, s + \frac{t_1}{\alpha}\right) = \Psi_1(x(s), t_1(s), t_2) - g(x(s), t_1(s), t_2).$$

Hence, from (70), we infer that

$$\Psi_1(s) = g(x, t_1, t_2) - g(x(s), t_1(s), t_2) + \Psi_1(x(s), t_1(s), t_2),$$

and therefore

$$\Psi_1(s) = \Psi_1\left(x + s\frac{(1-r)(2-r)}{2(3-r)}\dot{\alpha}(s\alpha + 2t_1), t_1 + s\alpha, t_2\right) - s\frac{1}{3-r}\dot{\alpha}x - s\frac{(2-r)^2}{2(3-r)^2}\ddot{\alpha}t_1^2$$
$$- s^2\frac{2-r}{6(3-r)^2}((1-r)\dot{\alpha}^2 + (2-r)\alpha\ddot{\alpha})(3t_1 + s\alpha). \tag{72}$$

Hence, given any solution $\Psi_1(x, t_1, t_2)$ of the potential r-dmKP equation (30) and any analytic function $\alpha(t_2)$ then

$$\tilde{\Psi}_1 = \Psi_1\left(x + \frac{(1-r)(2-r)}{2(3-r)}\dot{\alpha}(\alpha + 2t_1), t_1 + \alpha, t_2\right)$$
$$- \frac{1}{3-r}\dot{\alpha}x - \frac{(2-r)^2}{2(3-r)^2}\ddot{\alpha}t_1^2 - \frac{2-r}{6(3-r)^2}((1-r)\dot{\alpha}^2 + (2-r)\alpha\ddot{\alpha})(3t_1 + \alpha) \tag{73}$$

is a new solution of (30).

For $u = -\Psi_{1,x}$, thus u solves the r-dmKP equation 32; the corresponding symmetry transformation is given by

$$\tilde{u}(x, t_1, t_2) = u\left(x + \frac{(1-r)(2-r)}{2(3-r)}\dot{\alpha}(t_2)(\alpha(t_2) + 2t_1), t_1 + \alpha(t_2), t_2\right) + \frac{1}{3-r}\dot{\alpha}(t_2),$$

which for $r = 1$ simplifies to

$$\tilde{u}(x, t_1, t_2) = u(x, t_1 + \alpha(t_2), t_2) + \tfrac{1}{2}\dot{\alpha}(t_2).$$

3. Finally, we tackle the most involved case

$$\begin{aligned} F &= \alpha(t_2 + \varepsilon)L^{3-r} \\ &= \alpha(t_2)L^{3-r} + \frac{2-r}{3-r}\dot{\alpha}(t_2)t_1L^{2-r} + \left(\frac{1}{3-r}\dot{\alpha}(t_2)x + \frac{(2-r)^2}{2(3-r)^2}\ddot{\alpha}(t_2)t_1^2\right)L^{1-r} \\ &\quad + \left(-\frac{r}{3-r}\dot{\alpha}(t_2)\Psi_1 + \frac{2-r}{(3-r)^2}\ddot{\alpha}(t_2)t_1x + \frac{(2-r)^3}{6(3-r)^3}\dddot{\alpha}(t_2)t_1^3\right)L^{-r} + \cdots \end{aligned}$$

so that

$$\begin{aligned} \frac{\mathrm{d}\psi_<}{\mathrm{d}s}\cdot\psi_<^{-1} &= \alpha(t_2)\frac{\mathrm{d}\psi_<}{\mathrm{d}t_2}\cdot\psi_<^{-1} + \frac{2-r}{3-r}\dot{\alpha}(t_2)t_1\frac{\mathrm{d}\psi_<}{\mathrm{d}t_1}\cdot\psi_<^{-1} \\ &\quad - \left(\frac{1}{3-r}\dot{\alpha}(t_2)x + \frac{(2-r)^2}{2(3-r)^2}\ddot{\alpha}(t_2)t_1^2\right)P_<L^{1-r} - \left(-\frac{r}{3-r}\dot{\alpha}(t_2)\Psi_1\right. \\ &\quad \left. + \frac{2-r}{(3-r)^2}\ddot{\alpha}(t_2)t_1x + \frac{(2-r)^3}{6(3-r)^3}\dddot{\alpha}(t_2)t_1^3\right)L^{-r} + \cdots. \end{aligned} \tag{74}$$

From (74) we deduce that $\Psi_1(s)$ solves the following PDE:

$$\begin{aligned} \Psi_{1,s} &= \alpha\Psi_{1,t_2} + \frac{2-r}{3-r}\dot{\alpha}t_1\Psi_{1,t_1} + (1-r)\left(\frac{1}{3-r}\dot{\alpha}x + \frac{(2-r)^2}{2(3-r)^2}\ddot{\alpha}t_1^2\right)\Psi_{1,x} \\ &\quad + \frac{r}{3-r}\dot{\alpha}\Psi_1 - \frac{2-r}{(3-r)^2}\ddot{\alpha}t_1x - \frac{(2-r)^3}{6(3-r)^3}\dddot{\alpha}\,t_1^3. \end{aligned} \tag{75}$$

The general solution of (75) is

$$\Psi_1(s) = \alpha^{-r/(3-r)} f\Bigg(t_1\alpha^{-(2-r)/(3-r)}, x\alpha^{-(1-r)/(3-r)} - \frac{(1-r)(2-r)^2}{2(3-r)^2} t_1^2 \alpha^{-2(2-r)/(3-r)}\dot{\alpha}, s + \int^{t_2} \frac{\mathrm{d}t}{\alpha(t)}\Bigg) + g(x, t_1, t_2) \tag{76}$$

where f is an arbitrary function and g is given by

$$g(x, t_1, t_2) := \frac{2-r}{(3-r)^2}\frac{\dot{\alpha}}{\alpha} x t_1 + \frac{(2-r)^3}{6(3-r)^4}\left((3-r)\frac{\ddot{\alpha}}{\alpha} - (3-2r)\frac{\dot{\alpha}^2}{\alpha^2}\right) t_1^3.$$

We define

$$c(t) = \int^t \frac{\mathrm{d}t}{\alpha(t)}$$

and define T by the relation

$$c(T) = s + c(t_2)$$

or

$$\int_{t_2}^{T} \frac{\mathrm{d}t}{\alpha(t)} = s. \tag{77}$$

Observe that from (77) we derive

$$\frac{\mathrm{d}T}{\mathrm{d}t_2}\frac{1}{\alpha(T)} - \frac{1}{\alpha(t_2)} = 0 \Rightarrow \frac{\mathrm{d}T}{\mathrm{d}t_2} = \frac{\alpha(T(t_2))}{\alpha(t_2)} \tag{78}$$

and hence

$$\frac{\mathrm{d}^2 T}{\mathrm{d}t_2^2} = \frac{\mathrm{d}T}{\mathrm{d}t_2}\frac{\dot{\alpha}(T) - \dot{\alpha}(t_2)}{\alpha(t_2)}. \tag{79}$$

Now, we can write (76) as

$$\Psi_1(s) = \alpha^{-r/(3-r)} f\Bigg(t_1\alpha^{-(2-r)/(3-r)}, x\alpha^{-(1-r)/(3-r)} - \frac{(1-r)(2-r)^2}{2(3-r)^2} t_1^2 \alpha^{-2(2-r)/(3-r)}\dot{\alpha}, c(T)\Bigg) + g(x, t_1, t_2),$$

which setting $s = 0$ reads

$$\Psi_1 = \alpha^{-r/(3-r)} f\Bigg(t_1\alpha^{-(2-r)/(3-r)}, x\alpha^{-(1-r)/(3-r)} - \frac{(1-r)(2-r)^2}{2(3-r)^2} t_1^2 \alpha^{-2(2-r)/(3-r)}\dot{\alpha}, c(t_2)\Bigg) + g(x, t_1, t_2).$$

Then, we introduce the following curves:

$$t_2(s) := T(t_2), \qquad t_1(s) := \left(\frac{\alpha(t_2)}{\alpha(T(t_2))}\right)^{-(2-r)/(3-r)} t_1,$$

$$x(s) := \left(\frac{\alpha(t_2)}{\alpha(T(t_2))}\right)^{-(1-r)/(3-r)} x + \frac{(1-r)(2-r)^2}{2(3-r)^2}\alpha(T(t_2))^{(1-r)/(3-r)}\alpha(t_2)^{-2(2-r)/(3-r)}(\dot{\alpha}(T(t_2)) - \dot{\alpha}(t_2))t_1^2$$

which using (78) and (79) can be expressed as

$$\begin{aligned} &t_2(s) := T(t_2), \qquad t_1(s) := (\dot{T}(t_2))^{(2-r)/(3-r)} t_1, \\ &x(s) := (\dot{T}(t_2))^{(1-r)/(3-r)} \left(x + \frac{(1-r)(2-r)^2}{2(3-r)^2} \frac{\ddot{T}(t_2)}{\dot{T}(t_2)} t_1^2 \right). \end{aligned} \tag{80}$$

With the curve parametrized as in (80) we see that

$$\begin{aligned} &f\left(t_1 \alpha^{-(2-r)/(3-r)}, x\alpha^{-(1-r)/(3-r)} - \frac{(1-r)(2-r)^2}{2(3-r)^2} t_1^2 \dot{\alpha}, s + \int^{t_2} \frac{\mathrm{d}t}{\alpha(t)} \right) \\ &\quad = \alpha(T)^{r/(3-r)} (\Psi_1(x(s), t_1(s), t_2(s)) - g(x(s), t_1(s), t_2(s))), \end{aligned}$$

and therefore

$$\Psi_1(s) = g(x, t_1, t_2) - \dot{T}^{r/(3-r)} g(x(s), t_1(s), t_2(s)) + \dot{T}^{r/(3-r)} \Psi_1(x(s), t_1(s), t_2(s)). \tag{81}$$

Let us evaluate

$$g(x, t_1, t_2) - \dot{T}^{r/(3-r)} g(x(s), t_1(s), t_2(s)) = A(t_2) x t_1 + B(t_2) t_1^3,$$

where

$$A := \frac{2-r}{(3-r)^2} \left(\frac{\dot{\alpha}}{\alpha} - \dot{T} \frac{\dot{\alpha}(T)}{\alpha(T)} \right), \tag{82}$$

$$\begin{aligned} B := &\frac{(2-r)^3}{6(3-r)^4} \left((3-r) \left(\frac{\ddot{\alpha}}{\alpha} - \dot{T} \frac{\ddot{\alpha}(T)}{\alpha(T)} \right) - (3-2r) \left(\frac{\dot{\alpha}^2}{\alpha^2} - \dot{T}^2 \frac{\dot{\alpha}(T)^2}{\alpha(T)^2} \right) \right) \\ &- \frac{(2-r)^3}{2(3-r)^4} (1-r) \ddot{T} \frac{\dot{\alpha}(T)}{\alpha(T)}. \end{aligned} \tag{83}$$

From (78), (79) and (82) we derive

$$A = -\frac{2-r}{(3-r)^2} \frac{\ddot{T}}{\dot{T}}. \tag{84}$$

A similar expression may be derived, using (78) and (79) and its consequences, for B solely in terms of T and its derivatives. However, a faster way is to reckon $Axt_1 + Bt_1^3$ as a solution of (30) just by applying the symmetry to $\Psi_1 = 0$. In doing so we find that

$$B = -\frac{(2-r)^2}{6(3-r)} (\dot{A} + r(3-r)A^2)$$

and, hence, by taking into account (84) we deduce the following expression for B in terms solely of T and its derivatives:

$$B = -\frac{(2-r)^3}{6(3-r)^3} \left(\frac{\dddot{T}}{\dot{T}} - \frac{3}{3-r} \frac{\ddot{T}^2}{\dot{T}^2} \right). \tag{85}$$

Collecting (81) together with (80), (84) and (85) we deduce the following: If $\Psi(x, t_1, t_2)$ is a solution of the potential r-dmKP equation (30) and $T(t_2)$ is an arbitrary function of t_2 then

$$\begin{aligned} \tilde{\Psi}_1(x, t_1, t_2) = &-\frac{2-r}{(3-r)^2} \frac{\ddot{T}}{\dot{T}} x t_1 - \frac{(2-r)^3}{6(3-r)^3} \left(\frac{\dddot{T}}{\dot{T}} - \frac{3}{3-r} \frac{\ddot{T}^2}{\dot{T}^2} \right) t_1^3 \\ &+ \dot{T}^{r/(3-r)} \Psi_1 \left(\dot{T}^{(1-r)/(3-r)} \left(x + \frac{(1-r)(2-r)^2}{2(3-r)^2} \frac{\ddot{T}}{\dot{T}} t_1^2 \right), \dot{T}^{(2-r)/(3-r)} t_1, T \right) \end{aligned} \tag{86}$$

is a new solution of (30). As previously for $u = -\Psi_{1,x}$ we have: given a solution u of the r-dmKP equation (32) and an arbitary function $T(t_2)$ then $\tilde{u}$ is defined as

$$\tilde{u} = \frac{2-r}{(3-r)^2}\frac{\ddot{T}}{\dot{T}}x + \dot{T}^{1/(3-r)}u\left(\dot{T}^{(1-r)/(3-r)}\left(x + \frac{(1-r)(2-r)^2}{2(3-r)^2}\frac{\ddot{T}}{\dot{T}}t_1^2\right), \dot{T}^{(2-r)/(3-r)}t_1, T\right) \tag{87}$$

is a new solution of (32).

We collect these results regarding the potential r-dmKP equation in the following:

Proposition 5. *Given a solution Ψ_1 of the potential r-dmKP equation*

$$\Psi_{1,xt_2} = \frac{3-r}{(2-r)^2}\Psi_{1,t_1t_1} - \frac{(3-r)(1-r)}{2-r}\Psi_{1,xx}\Psi_{1,t_1} - \frac{(3-r)r}{2-r}\Psi_{1,x}\Psi_{1,xt_1}$$
$$- \frac{(3-r)(1-r)}{2}\Psi_{1,x}^2\Psi_{1,xx}$$

and arbitrary functions $\alpha(t_2)$, $T(t_2)$, the following functions are new solutions of the r-dmKP equation:

$$\tilde{\Psi}_1 = -\frac{2-r}{3-r}\dot{\alpha}(t_2)t_1 + \Psi_1(x + (1-r)\alpha(t_2), t_1, t_2),$$

$$\tilde{\Psi}_1 = -\frac{1}{3-r}\dot{\alpha}(t_2)x - \frac{(2-r)^2}{2(3-r)^2}\ddot{\alpha}(t_2)t_1^2 - \frac{2-r}{6(3-r)^2}((1-r)\dot{\alpha}(t_2)^2$$
$$+ (2-r)\alpha(t_2)\ddot{\alpha}(t_2))(3t_1 + \alpha(t_2))$$
$$+ \Psi_1\left(x + \frac{(1-r)(2-r)}{2(3-r)}\dot{\alpha}(t_2)(\alpha(t_2) + 2t_1), t_1 + \alpha(t_2), t_2\right)$$

$$\tilde{\Psi}_1 = -\frac{2-r}{(3-r)^2}\frac{\ddot{T}(t_2)}{\dot{T}(t_2)}xt_1 - \frac{(2-r)^3}{6(3-r)^3}\left(\frac{\dddot{T}(t_2)}{\dot{T}(t_2)} - \frac{3}{3-r}\frac{\ddot{T}(t_2)^2}{\dot{T}(t_2)^2}\right)t_1^3 + \dot{T}(t_2)^{r/(3-r)}\Psi_1$$
$$\times\left(\dot{T}(t_2)^{(1-r)/(3-r)}\left(x + \frac{(1-r)(2-r)^2}{2(3-r)^2}\frac{\ddot{T}(t_2)}{\dot{T}(t_2)}t_1^2\right), \dot{T}(t_2)^{(2-r)/(3-r)}t_1, T(t_2)\right).$$

A similar proposition for the r-dmKP equation (32) follows.

Proposition 6. *Given a solution u of the r-dmKP equation*

$$u_{t_2} = \frac{3-r}{(2-r)^2}(\partial_x^{-1}u)_{t_1t_1} + \frac{(3-r)(1-r)}{2-r}u_x(\partial_x^{-1}u)_{t_1}$$
$$+ \frac{r(3-r)}{2-r}uu_{t_1} - \frac{(3-r)(1-r)}{2}u^2u_x$$

and arbitrary functions $\alpha(t_2)$, $T(t_2)$, the following functions are new solutions of the r-dmKP equation:

$$\tilde{u} = u(x + (1-r)\alpha(t_2), t_1, t_2),$$

$$\tilde{u} = \frac{1}{3-r}\dot{\alpha}(t_2) + u\left(x + \frac{(1-r)(2-r)}{2(3-r)}\dot{\alpha}(t_2)(\alpha(t_2)+2t_1), t_1 + \alpha(t_2), t_2\right),$$

$$\tilde{u} = \frac{2-r}{(3-r)^2}\frac{\ddot{T}(t_2)}{\dot{T}(t_2)}t_1 + \dot{T}(t_2)^{1/(3-r)}u$$
$$\times\left(\dot{T}(t_2)^{(1-r)/(3-r)}\left(x + \frac{(1-r)(2-r)^2}{2(3-r)^2}\frac{\ddot{T}(t_2)}{\dot{T}(t_2)}t_1^2\right), \dot{T}(t_2)^{(2-r)/(3-r)}t_1, T(t_2)\right).$$

4.4. Symmetries of the potential r-dDym equation

We shall find three symmetries for the r-dDym equation (40). From (62) we deduce that

$$\frac{\mathrm{d}\bar{\psi}_{1-r}}{\mathrm{d}s}\cdot\bar{\psi}_{1-r}^{-1} + \mathrm{Ad}_{\bar{\psi}_{1-r}}P_{1-r}F(L,M) = 0,$$

where

$$\frac{\mathrm{d}\bar{\psi}_{1-r}}{\mathrm{d}s}\cdot\bar{\psi}_{1-r}^{-1} = \begin{cases}\dfrac{1}{1-r}\dfrac{\bar{X}_s}{\bar{X}_x}p^{1-r}, & r\neq 1,\\ -\xi_s p^{1-r}, & r = 1.\end{cases}$$

As for the previous case we pay particular attention to the case $N = 2$, with $n = 1-r,\ 2-r$ and $3-r$, so that F can be written as

$$\alpha\left(\frac{M}{(3-r)L^2}\right)L^{1-r},\qquad \alpha\left(\frac{M}{(3-r)L^2}\right)L^{2-r}\qquad\text{and}\qquad \alpha\left(\frac{M}{(3-r)L^2}\right)L^{3-r},$$

and

$$\alpha\left(\frac{M}{(3-r)L^2}\right) = \alpha(t_2) + \frac{2-r}{3-r}\dot{\alpha}(t_2)t_1L^{-1} + \left(\frac{1}{3-r}\dot{\alpha}(t_2)x + \frac{(2-r)^2}{2(3-r)^2}\ddot{\alpha}(t_2)t_1^2\right)L^{-2}$$
$$+\left(-\frac{r}{3-r}\dot{\alpha}(t_2)\Psi_1 + \frac{2-r}{(3-r)^2}\ddot{\alpha}(t_2)t_1x + \frac{(2-r)^3}{6(3-r)^3}\dddot{\alpha}(t_2)t_1^3\right)L^{-3} + \cdots.$$

Proceeding as in the $r$dmKP case we derive (for a complete proof see [29]).

Proposition 7. *Given a solution $\bar{X}$ of the potential r-dDym equation* ($r\neq 1$)

$$\bar{X}_{xt_2} = \frac{3-r}{2-r}\left(\frac{1}{2-r}\bar{X}_{t_1t_1}\bar{X}_x - \frac{1}{1-r}\bar{X}_{xt_1}\bar{X}_{t_1}\right),$$

and arbitrary functions $\alpha(t_2)$, $T(t_2)$, the following functions are new solutions of the potential r-dDym equation ($r \neq 1$):

$$\tilde{\bar{X}} = \bar{X}(x, t_1, t_2) - (1-r)\alpha(t_2),$$

$$\tilde{\bar{X}} = -\frac{(1-r)(2-r)}{2(3-r)}\dot{\alpha}(t_2)(\alpha(t_2) + 2t_1) + \bar{X}(x, t_1 + \alpha(t_2), t_2),$$

$$\tilde{\bar{X}} = -\frac{(1-r)(2-r)^2}{2(3-r)^2} t_1^2 \frac{\ddot{T}(t_2)}{\dot{T}(t_2)} + \dot{T}(t_2)^{-(1-r)/(3-r)} \bar{X}(x, \dot{T}(t_2)^{(2-r)(3-r)} t_1, T(t_2)).$$

Given a solution ξ of the potential $r = 1$ dDym equation

$$\xi_{t_2 x} - 2\xi_{t_1 t_1} - 2\xi_{t_1}\xi_{t_1 x} = 0$$

new solutions $\tilde{\xi}$ are given by

$$\tilde{\xi} = \alpha(t_2) + \xi(x, t_1, t_2),$$

$$\tilde{\xi} = \frac{1}{4}\dot{\alpha}(t_2)(\alpha(t_2) + 2t_1) + \xi(x, t_1 + \alpha(t_2), t_2),$$

$$\tilde{\xi} = \frac{x}{2}\log(\dot{T}(t_2)) + \frac{t_1^2}{8}\frac{\ddot{T}(t_2)}{\dot{T}(t_2)} + \xi\left(x, \sqrt{\dot{T}(t_2)}t_1, T(t_2)\right).$$

We also resume the results for the r-dDym equation (43).

Proposition 8. *Given a solution v of the r-dDym equation*

$$v_{t_2} = \frac{3-r}{(2-r)^2} v^{r-1}(v^{2-r}\partial_x^{-1}(v^{r-2}v_{t_1}))_{t_1},$$

and arbitrary functions $\alpha(t_2)$, $T(t_2)$, the following functions are new solutions of the r-dDym equations:

$$\tilde{v} = v(x, t_1 + \alpha(t_2), t_2), \qquad \tilde{v} = \dot{T}(t_2)^{1/(3-r)} v(x, \dot{T}(t_2)^{(2-r)/(3-r)} t_1, T(t_2)).$$

4.5. Symmetries of the r-dToda equation

The r-dToda equation (52*a*) mixes the independent variables x, t_1 and $\bar{t}_1$. Let us analyse the symmetries associated with the t_1-flow, i.e. study the action of additional symmetries generated by $F(L, M)$. Suppose that $N = 1$, then we have the cases $n = 1 - r$ and $2 - r$, so that we have two different generators, namely

$$\alpha\left(\frac{M}{(2-r)L}\right)L^{1-r} \qquad \text{and} \qquad \alpha\left(\frac{M}{(2-r)L}\right)L^{2-r}.$$

There are also additional symmetries associated with the $\bar{t}_1$ flow. Now $\bar{N} = 1$, $n = -r$ and $n = 1 - r$ and there are two different generators:

$$\bar{\alpha}\left(\frac{\bar{M}}{-r\bar{L}^{-1}}\right)\bar{L}^{1-r} \qquad \text{and} \qquad \bar{\alpha}\left(\frac{\bar{M}}{-r\bar{L}^{-1}}\right)\bar{L}^{-r}.$$

We have the following (for a complete proof see [29]).

Proposition 9. *Given a solution $\bar{X}$ of the r-dToda equation*

$$((\bar{X}_x)^{-(2-r)/(1-r)})_x - \frac{1}{(1-r)r}\left(\frac{\bar{X}_{\bar{t}_1}}{\bar{X}_x}\right)_{t_1} = 0,$$

and arbitrary functions $\alpha(t_1)$, $T(t_1)$, $\bar{\alpha}(\bar{t}_1)$ and $\bar{T}(\bar{t}_1)$, the following functions are new solutions of the r-dToda equation:

$$\tilde{\bar{X}} = \bar{X}(x, t_1, \bar{t}_1) - (1-r)\alpha(t_1), \qquad \tilde{\bar{X}} = \dot{T}(t_1)^{-(1-r)/(2-r)}\bar{X}(x, T(t_1), \bar{t}_1),$$

$$\tilde{\bar{X}} = \bar{X}(x + (1-r)\bar{\alpha}(\bar{t}_1), t_1, \bar{t}_1), \qquad \tilde{\bar{X}} = \bar{X}(\dot{\bar{T}}(\bar{t}_1)^{-(1-r)/r}x, t_1, \bar{T}(\bar{t}_1)).$$

Finally, observe that the symmetries derived for the $r = 1$ case, i.e. the Boyer–Finley equation $(\exp(\xi_x))_x + \xi_{t_1\bar{t}_1} = 0$, the symmetries are the well known, since 1986 by P Olver, conformal symmetries of the dToda equation.

4.6. Some solutions

Now we study the t_2 invariance on the potential of r-dDym equation (40) and using the Miura type map, corresponding to solutions for the t_2 invariant solutions of the r-dmKP equation (32), which we may call r dispersionless modified Boussinesq equation. We analyse here solutions of the (40) which do not depend on one of the variables t_2. Thus, (40) simplifies to

$$(1-r)\bar{X}_{t_1t_1}\bar{X}_x = (2-r)\bar{X}_{xt_1}\bar{X}_{t_1}$$

which can be written as

$$(\log(\bar{X}_{t_1}^{1-r}))_{t_1} = (\log(\bar{X}_x^{2-r}))_{t_1}$$

so that

$$\left(\log\left(\frac{\bar{X}_{t_1}^{1-r}}{\bar{X}_x^{2-r}}\right)\right)_{t_1} = 0.$$

This last equation is equivalent to

$$\frac{1}{k'(x)}\bar{X}_x = \bar{X}_{t_1}^{(1-r)/(2-r)},$$

where k' is the derivative of $k(x)$, an arbitrary function of x. Thus, if we introduce the variable $\tilde{x} = k(x)$ we have

$$\bar{X}_{\tilde{x}} = \bar{X}_{t_1}^{(1-r)/(2-r)}.$$

This is a first-order nonlinear (for $1 - r \neq 0$) PDE, and we will solve it by the the method of the complete solution. First observe that a complete integral is $\bar{X}(\tilde{x}, t_1; a, b) = a^{(1-r)/(2-r)}\tilde{x} + at_1 + b$. Then, the corresponding envelope is given by

$$\bar{X}(x, t_1) = a(k(x), t_1)^{(1-r)/(2-r)}k(x) + a(k(x), t_1)t_1 + f(a(k(x), t_1)$$

which depends on two arbitrary functions k and f on one variable, being therefore a general solution. For example, if $f = 0$, by applying the symmetries given in proposition 7, we have the following new solutions:

$$\tilde{\bar{X}} = \frac{\kappa(x)^{2-r}}{t_1^{1-r}} - (1-r)\alpha(t_2),$$

$$\tilde{\bar{X}} = -\frac{(1-r)(2-r)}{2(3-r)}\dot{\alpha}(t_2)(\alpha(t_2) + 2t_1) + \frac{\kappa(x)^{2-r}}{(t_1 + \alpha(t_2))^{1-r}},$$

$$\tilde{\bar{X}} = -\frac{(1-r)(2-r)^2}{2(3-r)^2}t_1^2\frac{\ddot{T}(t_2)}{\dot{T}(t_2)} + \frac{\kappa(x)^{2-r}}{(\dot{T}(t_2)t_1)^{1-r}}.$$

The second and third families are nontrivial solutions depending on the two arbitrary functions of one variable each. Finally, another example is given by $f(a) = -(2-r)/(1+r)a^{(2-r)/(1+r)}$; by introducing the functions

$$F := \sqrt[3]{108t_1 + 12\sqrt{3}\sqrt{4\frac{(1-r)^3}{(2-r)^3}\tilde{x}^3 + t_1^2}}, \qquad \alpha := -\frac{1}{6}F + 2\frac{1-r}{2-r}\frac{\tilde{x}}{F}$$

a solution is

$$\bar{X} = \left(\frac{\tilde{x}}{\alpha} + \frac{t_1}{\alpha^2} - \frac{2-r}{1+r}\alpha\right)\alpha^r.$$

We shall use the Miura map

$$-\frac{1}{(1-r)(2-r)}\bar{X}_{t_1}(X(x,t_1,t_2),t_1,t_2) = u(x,t_1,t_2)$$

to get a solution of the r-dmKP equation. The corresponding solution of the r-dmKP equation (32)

$$u = \frac{1}{2-r}\frac{x}{t_1} \tag{88}$$

for the potential r-dmKP, and after applying the three master symmetries described above we get the following solutions:

$$\tilde{\Psi}_1 = -\frac{2-r}{3-r}\dot{\alpha}(t_2)t_1 - \frac{1}{2(2-r)}\frac{(x+(1-r)\alpha(t_2))^2}{t_1},$$

$$\tilde{\Psi}_1 = -\frac{1}{3-r}\dot{\alpha}(t_2)x - \frac{(2-r)^2}{2(3-r)^2}\ddot{\alpha}(t_2)t_1^2 - \frac{1}{2(2-r)}\frac{\left(x+\frac{(1-r)(2-r)}{2(3-r)}\dot{\alpha}(t_2)(\alpha(t_2)+2t_1)\right)^2}{t_1+\alpha(t_2)}$$
$$- \frac{2-r}{6(3-r)^2}((1-r)\dot{\alpha}(t_2)^2 + (2-r)\alpha(t_2)\ddot{\alpha}(t_2))(3t_1+\alpha(t_2)),$$

$$\tilde{\Psi}_1 = -\frac{2-r}{2(3-r)}\frac{\ddot{T}(t_2)}{\dot{T}(t_2)}xt_1 - \frac{(2-r)^3}{2(3-r)^3}\left(\frac{1}{3}\frac{\dddot{T}(t_2)}{\dot{T}(t_2)} - \frac{1+r}{4}\frac{\ddot{T}(t_2)^2}{\dot{T}(t_2)^2}\right)t_1^3 - \frac{1}{2(2-r)}\frac{x^2}{t_1}.$$

5. Twistor equations

Previously, we have introduced the Lax and Orlov functions as the following canonical transformations of the pair p, x:

$$L = \mathrm{Ad}_{\psi_<\cdot\exp t}p, \qquad \bar{\ell} = \mathrm{Ad}_{\psi_>\cdot\exp\bar{t}}p, \qquad \bar{L} = \mathrm{Ad}_{\psi_\geqslant\cdot\exp\bar{t}}p,$$
$$M := \mathrm{Ad}_{\psi_<\cdot\exp t}x, \qquad \bar{m} := \mathrm{Ad}_{\psi_>\cdot\exp\bar{t}}x, \qquad \bar{M} := \mathrm{Ad}_{\psi_\geqslant\cdot\exp\bar{t}}x.$$

Thus, they satisfy

$$\{L, M\} = L^r, \qquad \{\bar{\ell}, \bar{m}\} = \bar{\ell}^r, \qquad \{\bar{L}, \bar{M}\} = \bar{L}^r.$$

Other important functions are

$$P := \mathrm{Ad}_h\, p, \qquad Q := \mathrm{Ad}_h\, x, \qquad \bar{P} := \mathrm{Ad}_{\bar{h}}\, p, \qquad \bar{Q} := \mathrm{Ad}_{\bar{h}}\, x$$

for which we have

$$\{P, Q\} = P^r, \qquad \{\bar{P}, \bar{Q}\} = \bar{P}^r.$$

These functions result from the canonical transformation of the p, x variables generated by the initial conditions $h, \bar{h}$ of the factorization problem (4).

It is easy to prove that

Proposition 10. *For any solution $\psi_<$ and $\psi_\geqslant$ of the factorization problem (4) the following twistor equations hold:*

$$P(L, M) = \bar{P}(\bar{L}, \bar{M}), \qquad Q(L, M) = \bar{Q}(\bar{L}, \bar{M}). \tag{89}$$

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